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ON CONVERGENCE OF PAL-TYPE INTERPOLATION POLYNOMIALS

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Abstract

Let $\{x_k^*\}_{k=1}^{n-1}$ be the zeros of the $(n-1)$-th Legendre polynomial $p_{n-1}(x)$ and $\{x_k\}_{k=1}^{n}$ be the zeros of the polynomial $w(x)=(1-x^2)p_{n-1}^1(x)$. By the theory of the Pal interpolation, for a function $f\in C_{[-1,1]}^1$, there exists a unique polynomial $Q_n(f, x)$ of degree $2n-1$ satisfying conditions $Q_n(f, x_k)=f(x_k)$, $Q_n'(f, x_k^*)=f'(x_k^*)$, where $k=1, 2, \cdots, n$ and $x_n^*=-1$. The main result of this paper is that if $f\in C_{[-1,1]}^r$, then

$$f(x)-Q_n(f, x)=O(1)W(x)w\left(f^{(r)}, \frac{1}{n}\right)n^{\frac{1}{2}-r}, \quad -1\leqslant x\leqslant 1.$$

Hence, if $f\in C_{[-1,1]}^1$, then $Q_n(f, x)$ converges to the function $f(x)$ uniformly on the interval $[-1, 1]$.

§1. Introduction

Let

$$-1=x_n<x_{n-1}<\cdots<x_2<x_1=1$$

be the zeros of the polynomial

$$W_n(x)=-n(n-1)\int_{-1}^{x}p_{n-1}(t)dt=(1-x^2)p_{n-1}'(x), \tag{1.1}$$

where $p_{n-1}(x)$ is the $(n-1)$-th Legendre polynomial with the usual normalization $p_{n-1}(1)=1$. The zeros of the derivative $W_n'(x)=-n(n-1)p_{n-1}(x)$ are denoted by x_k^* $(k=1, 2, \cdots, n-1)$. It is easy to see that

$$-1=x_n<x_{n-1}^*<x_{n-2}<\cdots<x_2<x_1^*<x_1=1. \tag{1.2}$$

Let f(x) be r-times continuously differentiable on $[-1, 1]$, that is $f\in C_{[-1,1]}^r$ and $r\geqslant 1$. By the general theory of Pal interpolation (see [1]), there exists a unique polynomial $Q_n(f, x)$ of degree $2n-1$ satisfying the following conditions:

$$Q_n(f, x_k)=f(x_k) \quad \text{and} \quad Q_n'(f, x_k^*)=f'(x_k),$$

where $k=1, 2, \cdots, n$ and $x_n^*=-1$. Hence, the polynomial can be called Pal-type interpolation based on the nodes (1.2).

Recently, S. A. N. Eneduanya[2] pointed out the polynomial $Q_n(f, x)$ can be represented in the form

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$$Q_n(f, x)=\sum_{k=1}^{n} f(x_k)A_k(x)+\sum_{k=1}^{n} f'(x_k^*)B_k(x)+C_n(x)f'(-1), \tag{1.3}$$

where $A_k(x)$, $B_k(x)$ and $C_n(x)$ are polynomials of degree $2n-1$. Let

$$l_k(x)=\frac{W_n(x)}{W'_n(x_k)(x-x_k)} \quad\text{and}\quad l_k^*(x)=\frac{W'_n(x)}{W''_n(x_k^*)(x-x_k^*)}. \tag{1.4}$$

Then

$$A_k(x)=l_k^2(x)-\frac{2W_n(x)}{W'_n(x_k)}\int_{-1}^{x}\frac{l'_k(t)}{t-x_k}dt \quad (k=2, 3, \cdots, n-1), \tag{1.5}$$

$$B_k(x)=\frac{W_n(x)}{W_n(x_k^*)}\int_{-1}^{x} l_k^*(t)dt \quad (k=1, 2, \cdots, n-1), \tag{1.6}$$

$$A_1(x)=l_1^2(x)-\frac{2W_k(x)}{W'_n(1)}\int_{-1}^{x}\frac{l'_1(t)+l'_1(1)p_{n-1}(t)}{t-1}dt,$$

$$C_n(x)=\frac{W_n(x)}{W'_n(-1)},$$

and

$$A_n(x)=\left(1-\frac{n(n-1)}{2}(x+1)\right)l_n^2(x)+\frac{W_n(x)}{W'_n(-1)}\int_{-1}^{x}\left\{\frac{n(n-1)}{2}l_n(t)\right.$$

$$\left.-2\left(1-\frac{n(n-1)}{2}(t+1)\right)l'_n(t)+\alpha p_{n-1}(t)\right\}\frac{dt}{t+1},$$

where

$$\alpha=2l'_n(-1)-\frac{n(n-1)}{2}l_n(-1).$$

On the convergence of $Q_n(f, x)$ as $n\to\infty$, Eneduanya[2] proved: If $f\in C^r_{[-1,1]}$ and $r\geqslant 1$, then for arbitrary $x\in[-1, 1]$ and $n\geqslant 2r+3$,

$$f(x)-Q_n(f, x)=O(1)w\left(f^{(r)}, \frac{1}{n}\right)n^{-r+\frac{3}{2}}\ln n. \tag{1.7}$$

The purpose of this article is to introduce a new method which gives an essential improvement of above estimate (1.7). The main result is the following theorem.

Theorem. *If $f\in C^r_{[-1,1]}$ and $r\geqslant 1$, then*

$$f(x)-Q_n(f, x)=O(1)\frac{|W_n(x)|}{\sqrt{n}}\omega\left(f^{(r)}, \frac{1}{n}\right)n^{1-r}$$

holds uniformly in $x\in[-1, 1]$, $n\geqslant 2r+3$.

Since $W_n(x)=O(1)\sqrt{n}$, the theorem implies the following corollary.

Corollary. *If $f\in C^1_{[-1,1]}$, then $Q_n(f, x)$ converges to $f(x)$ uniformly in the interval $[-1, 1]$.*

§ 2. Preliminaries

In order to prove the theorem, we state some properties of Legendre polynomial $p_{n-1}(t)$ as follows. By the well known estimations

$$(1-x^2)^{\frac{1}{4}}p_{n-1}(x)\sqrt{n}=O(1) \quad\text{and}\quad |p_{n-1}(x)|\leqslant 1, \tag{2.1}$$

and Bernstein's inequality (see [6]) we have

$$(1-x^2)^{\frac{3}{4}}p'_{n-1}(x)=O(1)\sqrt{n}. \tag{2.2}$$

The following relations can be found in[3]:

$$1-x_\nu^2\sim\sin^2\frac{\nu\pi}{n}\quad(\nu=2,\ 3,\ \cdots,\ n-1), \tag{2.3}$$

$$|p_{n-1}(x_\nu)|\sim\left(n\sin\frac{\nu\pi}{n}\right)^{-\frac{1}{2}}\quad(\nu=2,\ 3,\ \cdots,\ n-1). \tag{2.4}$$

Let $x_\nu^*=\cos\theta_\nu^*(0\leqslant\theta_\nu^*\leqslant\pi,\ \nu=1,\ 2,\ \cdots,\ n-1)$. Then

$$\frac{2\nu-1}{2n-1}\pi\leqslant\theta_\nu^*\leqslant\frac{2\nu}{2n-1}\pi \tag{2.5}$$

and

$$|p'_{n-1}(x_\nu^*)|\sim n^2\left(n\sin\frac{\nu\pi}{n}\right)^{-\frac{3}{2}}. \tag{2.6}$$

Here notation $a_{n\nu}\sim b_{n\nu}$ means that there exists a positive constant M such that

$$\frac{b_{n\nu}}{M}\leqslant a_{n\nu}\leqslant Mb_{n\nu}$$

hold for $n=1,\ 2,\ \cdots$ and $\nu=2,\ 3,\ \cdots,\ n-1$.

The proof of Theorem is based on the following lemmas.

Lemma 1. *For $k=2,\ 3,\ \cdots,\ n-1$ and $x\in[-1,\ 1]$, the ineqvality*

$$|A_k(x)|\leqslant5|l_k(x)|+\frac{2|W_k(x)|}{n(n-1)(1-x_k^2)|p_{n-1}(x_k)|^3} \tag{2.7}$$

holds.

Proof First let $x_k>x$. By (1.5) we have

$$|A_k(x)|\leqslant|l_k(x)|^2+\frac{2|W_n(x)|}{|W'_n(x_k)|}\left|\int_{-1}^{x}\frac{l'_k(t)}{t-x_k}dt\right|. \tag{2.8}$$

But (see[2])

$$|l_k(x)|\leqslant1 \tag{2.9}$$

and

$$\int_{-1}^{x}\frac{l'_k(t)}{t-x_k}dt=\frac{l_k(x)}{x-x_k}+\int_{-1}^{x}\frac{l_k(t)}{(t-x_k)^2}dt$$

imply

$$\left|\int_{-1}^{x}\frac{l'_k(t)}{t-x_k}dt\right|\leqslant\frac{2}{|x-x_k|}. \tag{2.10}$$

Hence, from (2.8) (2.9) and (2.10) it follows that

$$|A_k(x)|\leqslant5|l_k(x)|. \tag{2.11}$$

Now, we assume $x>x_k$. By using (see[3])

$$\int_{-1}^{1}\frac{l'_k(t)}{t-x_k}dt=\frac{1}{(1-x_k^2)|(p_{n-1}(x_k)|^2}$$

and

$$\int_{-1}^{x}\frac{l'_k(t)}{t-x_k}dt=\int_{-1}^{1}\frac{l'_k(t)}{t-x_k}dt-\int_{x}^{1}\frac{l'_k(t)}{t-x_k}dt,$$

we obtain

$$\left|\int_{-1}^{x}\frac{l'_k(t)}{t-x_k}dt\right|\leqslant\frac{1}{(1-x_k^2)|p_{n-1}(x_k)|^2}+\frac{2}{x-x_k}.$$

Hence (2.8) and (2.9) imply

$$|A_k(x)|\leqslant 5|l_k(x)|+\frac{2|W_k(x)|}{(1-x_k^2)|W_n'(x_k)||p_{n-1}(x_k)|^2}.$$

From (1.1) we then have

$$|A_k(x)|\leqslant 5|l_k(x)|+\frac{2|W_n(x)|}{n(n-1)(1-x_k^2)|p_{n-1}(x_k)|^3}. \tag{2.12}$$

Combining (2.11) and (2.12) we obtain (2.7) and complete the proof of Lemma 1.

Lemma 2. *For* $k=1, 2, \cdots, n-1,$ *and* $x\in[-1, 1]$, $x\neq x_k^*$, *the following inequality*

$$|B_k(x)|\leqslant\frac{2|W_n(x)|}{|W_n(x_k^*)p_{n-1}^1(x_k^*)|}\left\{\frac{1}{|W_n(x_k^*)|}+\frac{1}{|x-x_k^*|}\sup_{|t|\leqslant 1}\left|\int_{-1}^t p_{n-1}(u)du\right|\right\} \tag{2.13}$$

holds.

Proof According to the definitions (1.4) and (1.6), it is easy to see that for $x_k^*>x$,

$$B_k(x)=\frac{W_n(t)}{W_n(x_k^*)p_{n-1}^1(x_k^*)}\left\{\frac{1}{(x-x_k^*)}\int_{-1}^x p_{n-1}(t)dt+\int_{-1}^x\frac{1}{(t-x_k^*)^2}\int_{-1}^t p_{n-1}(u)du\,dt\right\}.$$

Hence we have

$$|B_k(x)|\leqslant\frac{2|W_k(x)|}{|W_n(x_k^*)p_{n-1}^1(x_k^*)|}\,\frac{1}{|x-x_k^*|}\sup_{-1<t\leqslant x}\left|\int_{-1}^t p_{n-1}(u)du\right|. \tag{2.14}$$

Now we assume $x>x_k^*$ and write

$$B_k(x)=\frac{W_n(x)}{W_n(x_k^*)p_{n-1}^1(x_k^*)}\left\{\int_{-1}^1\frac{p_{n-1}(t)}{t-x_k^*}dt-\int_x^1\frac{p_{n-1}(t)}{t-x_k^*}dt\right\}. \tag{2.15}$$

Since

$$\int_x^1\frac{p_{n-1}(t)}{t-x_k}dt=\frac{1}{x-x_k^*}\int_x^1 p_{n-1}(u)du+\int_x^1\frac{dt}{(t-x_k^*)^2}\int_t^1 p_{n-1}(u)du$$

and

$$\int_{-1}^1 p_{n-1}(u)du=0,$$

we have

$$\left|\int_x^1\frac{p_{n-1}(t)}{t-x_k^*}dt\right|\leqslant\frac{2}{|x-x_k^*|}\sup_{x\leqslant t\leqslant 1}\left|\int_1^t p_{n-1}(u)du\right|. \tag{2.16}$$

But (see [5])

$$\int_{-1}^1\frac{p_{n-1}(t)}{t-x_k^*}dt=\frac{1}{W_n(x_k^*)}. \tag{2.17}$$

Hence (2.15) and (2.16) imply

$$|B_k(x)|\leqslant\frac{2|W_n(x)|}{|W_n(x_k^*)p_{n-1}'(x_k^*)|}\left\{\frac{1}{|W_n(x_k^*)|}+\frac{1}{x-x_k^*}\sup_{x\leqslant t\leqslant 1}\left|\int_{-1}^t p_{n-1}(u)du\right|\right\}. \tag{2.18}$$

Thus from (2.14) and (2.18) it follows that (2.13) holds. Lemma 2 is proved.

§ 3. The Proof of Theorem

For the proof of Theorem we shall use a result of Gopengaus[7] which can be stated as follows: If $f\in C^r_{[-1,1]}$, then there exists a polynomial $g(x)$ of degree at most $(2n-1)\geqslant 4r+5$ such that for all $x\in[-1,1]$,

$$f^{(s)}(x)-g^{(s)}(x)=O(1)\,\omega\Big(f^{(r)},\frac{\sqrt{1-x^2}}{n}\Big)\Big(\frac{\sqrt{1-x^2}}{n}\Big)^{r-s}\quad (s=0,1,\cdots,r).\tag{3.1}$$

Obviously, $Q_n(g,x)=g(x)$. Hence (3.1) implies

$$\begin{aligned}Q_n(f,x)-f(x)&=Q_n(f-g,x)-(f(x)-g(x))\\&=\sum_{k=2}^{n-1}(f(x_k)-g(x_k)-f(x)+g(x))A_k(x)+\sum_{k=1}^{n-1}(f'(x_k^*)-g'(x_k^*))B_k(x).\end{aligned}$$

Therefore we have

$$|Q_n(f,x)-f(x)|\leqslant I_1+I_2,\tag{3.2}$$

where

$$I_1=\sum_{k=2}^{n-1}|f(x_k)-g(x_k)-f(x)+g(x)|\,|A_k(x)|$$

and

$$I_2=O(1)\,\omega\Big(f^{(r)},\frac{1}{n}\Big)\frac{1}{n^{r-1}}\sum_{k=1}^{n-1}|B_k(x)|.$$

Set

$$I_{11}=5\sum_{k=2}^{n-1}|f(x_k)-g(x_.)-f(x)+g(x)|\,|l_k(x)|,$$

$$I_{12}=\frac{2|W_n(x)|}{n(n-1)}\sum_{k=2}^{n-1}\frac{1}{(1-x_k^2)(p_{n-1}(x_k))|^3}|f(x_k)-g(x_k)-f(x)+g(x)|.$$

Then Lemma 1 implies

$$I_1\leqslant I_{1,1}+I_{1,2}.\tag{3.3}$$

Using (1.4) (2.4) and (3.1) we obtain

$$\begin{aligned}I_{11}&=O(1)\sum_{k=2}^{n-1}|l_k(x)|\,|x-x_k|f'(\xi)-g'(\xi)|\\&=O(1)\,\omega\Big(f^{(r)},\frac{1}{n}\Big)\frac{1}{n^{r-1}}\sum_{k=2}^{n-1}\frac{\Big(n\sin\frac{k\pi}{n}\Big)^{\frac{1}{2}}}{n^2}|W_k(x)|.\end{aligned}$$

So

$$I_{1,1}=O(1)\frac{|W_n(x)|}{\sqrt{n}}\,\omega\Big(f^{(r)},\frac{1}{n}\Big)n^{-r+1}.\tag{3.4}$$

On the other hand using (2.4) and (3.1) we get

$$\begin{aligned}I_{1,2}&=O(1)\,w\Big(f^{(r)},\frac{1}{n}\Big)\frac{1}{n^r}\sum_{k=2}^{n-1}\frac{|W_n(x)|}{n^2}\frac{\Big(n\sin\frac{k\pi}{n}\Big)^{\frac{3}{2}}}{\Big(\sin\frac{k\pi}{n}\Big)^2}\\&=O(1)\,\omega\Big(f^{(r)},\frac{1}{n}\Big)\frac{|W_n(x)|}{\sqrt{n}}\cdot n^{-r+1}.\end{aligned}$$

Hence combining (3.3) and (3.4) we have

$$I_1=O(1)\frac{|W_n(x)|}{\sqrt{n}}\,\omega\Big(f^{(r)},\frac{1}{n}\Big)n^{-r+1}.\tag{3.5}$$

In order to estimate I_2, we assume first that $x\geqslant 0$, and denote by x_j^* the zero of $W_n'(x)$ which is nearest to x. Writing

$$I_{2,1}=\sum_{k=1}^{j-1}|B_k(x)| \quad \text{and} \quad I_{2,2}=\sum_{k=j+1}^{n-1}|B_k(x)|,$$

we have

$$I_2=(I_{2,1}+I_{2,2}+(B_j(x)|)O(1)\omega\Big(f^{(r)},\frac{1}{n}\Big)\frac{1}{n^{r-1}}. \tag{3.6}$$

By (2.24), we get

$$I_{2,1}\leqslant 2|W_n(x)|\sum_{k=1}^{j-1}\frac{1}{|W_n(x_k^*)p_{n-1}^1(x_k^n)|}\frac{1}{|x-x_k^*|}\sup_{|t|\leqslant 1}\Big|\int_{-1}^{s}p_{n-1}(u)du\Big|.$$

Hence from (2.6) and (2.5) it follows that

$$I'_{2,1}=O(1)|W_n(x)|\sum_{k=1}^{j-1}\frac{1}{(1-x_k^2)|p'_{n-1}(x_k^*)|^2}\frac{1}{|x-x_k^*|}\frac{1}{n^2}\sup_{|t|\leqslant 1}|(1-t^2)p'_{n-1}(t)|$$

$$=O(1)|W_n(x)|\sum_{k=1}^{j-1}\frac{1}{(k+j)(j-k)}\frac{\sqrt{n}}{n^2}$$

and

$$I_{2,1}=O\frac{|W_n(x)|}{\sqrt{n}}. \tag{3.7}$$

Similarly we have

$$I_{2,2}=O(1)|W_n(x)|\sum_{k=j+1}^{n}\left\{\frac{\Big(n\sin\frac{k\pi}{n}\Big)^{\frac{q}{2}}}{\Big(\sin\frac{k\pi}{n}\Big)^4 n^6}+\frac{\Big(n\sin\frac{k\pi}{n}\Big)^3}{\Big(\sin\frac{k\pi}{n}\Big)^2 n^4}\frac{n^2}{(k+j)(k-j)}\frac{\sqrt{n}}{n^2}\right\}.$$

So

$$I_{2,2}=O(1)\frac{|W_n(x)|}{\sqrt{n}}. \tag{3.8}$$

Now we are going to estimate $B_j(x)$. First we assume that $x<x_1^*$. Obviously, (1.6) implies

$$B_j(x)=\frac{W_n(x)}{W_n(x_j^*)}\Big(\int_{-1}^{x_{j+1}^*}l_j^*(t)dt+\int_{x_{j+1}^*}^{x}l_j^*(t)dt\Big).$$

Hence from (1.4) and (2.6) we obtain

$$B_j(x)=O(1)\frac{|W_n(x)|}{|W_n(x_j^*)p'_{n-1}(x_j^*)|}\Big(\frac{1}{x_j^*-x_{j+1}^*}\sup_{|t|\leqslant 1}\Big|\int_{-1}^{t}p_{n-1}|u|du\Big|+\Big|\int_{x_{j+1}^*}^{x}\frac{p_{n-1}(t)}{t-x_j^n}dt\Big|\Big)$$

$$=O(1)|W_n(x)|(n^{-\frac{3}{2}}+jv^{-2}|x-x_{j+1}^*|\sup_{x_{j+1}^*<\xi\leqslant x}|p_{n-1}^1(\xi)|).$$

Therefore, by (2.5) and (2.2) we have

$$B_j(x)=O(1)|W_n(x)|\Big\{n^{-\frac{3}{2}}+\frac{j^2}{n^4}\frac{\sqrt{n}}{(\sqrt{1+x_j^{*2}})^{3/2}}\Big\}=O(1)\frac{|W_n(x)|}{\sqrt{n}}. \tag{3.9}$$

For the case $x>x_1^*$, we write

$$B_1(x)=\frac{W_n(x)}{W_n(x_1^*)}\Big\{\int_{-1}^{1}l_1^*(t)dt-\int_{x}^{1}l_1^*(t)dt\Big\},$$

then (2.17) and (1.4) imply

$$B_1(x)=\frac{W_n(x)}{W_n(x_1^*)}\Big(\frac{1}{|p_{n-1}^1(x_1^*)|^2(1-x_1^{*2})}-\int_x^1\frac{p_{n-1}(t)}{p_{n-1}^1(x_1^*)(t-x_1^*)}dt\Big).$$

Hence from (2.6) and the estimations

$$1-x=O(1)\frac{1}{n^2} \quad \text{and} \quad |p_n^1(t)| \leqslant n^2,$$

it follows that

$$B_1(x)=O(1)|W_n(x)|(n^{-2}+n^{-2}(1-x))\sup_{x<t\leqslant 1}|p_n'(t)|=O(1)|W_n(x)|(n^{-2}+n^{-4}\cdot n^2).$$

That is

$$B_1(x)=O(1)\frac{|W_n(x)|}{n^2}. \tag{3.10}$$

Combining (3.7)—(3.9) and (3.10) we have

$$\sum_{k=1}^{n-1}|B_k(x)|=O(1)\frac{|W_n(x)|}{\sqrt{n}} \quad (0\leqslant x\leqslant 1).$$

Thus

$$I_2=O(1)\frac{W_n(x)}{\sqrt{n}}\omega\left(f^{(r)}, \frac{1}{n}\right)n^{-r+1} \quad (0\leqslant x\leqslant 1).$$

From this and (3.2) (3.5) we can see that

$$Q_n(f, x)-f(x)=O(1)\frac{|W_n(x)|}{\sqrt{n}}\omega\left(f^{(r)}, \frac{1}{n}\right)n^{-r+1} \quad (0\leqslant x\leqslant 1).$$

Similarly we can prove that the above estimate holds also in interval [−1, 0]. This completes the proof of Theorem.

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