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OSCILLATION AND ASYMPTOTIC BEHAVIOR OF HIGHER ORDER NEUTRAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract

The authors establish sufficient conditions for the oscillation of all solution of neutral delay differential equations of even and odd order of the form

$$\frac{d^n}{dt^n}[y(t)+p(t)y(t-\tau)]+Q(t)y(t-\sigma)=0, \quad t\geqslant t_0,$$

where $P, Q\in C[[t_0, \infty), R]$, $\tau, \sigma\in R^+$ and $n\geqslant 1$.

§ 1. Introduction

A neutral delay differential equation (NDDE) is a differential equation in which the highest order derivative of the unknown function appears both with and without delays.

In general, the study of NDDEs presents complications which are unfamiliar to nonneutral differential equations with deviating arguments. For example, it has been shown by Snow[15] (see also [14]), that even though the characteristic roots of a NDDE may all have negative real parts, it is still possible for the equation to have unbounded solutions. Such behavior is impossible for nonneutral equations.

The study of the asymptotic and oscillatory behavior of the solutions of NDDEs, besides its theoretical interest, has some interest in applications. NDDEs appear, for example, in problems dealing with networks containing lossless transmission lines. Such networks arise in high speed computers where the lossless transmission lines are used to interconnect switching circuits[2, 9]. Neutral equations of order two appear in the study of vibrating masses attached to an elastic bar and also (as the Euler equation) in some variational problems[9].

Recently, Ladas and Sficas[10, 11], Grammatikopoulos, Grove and Ladas[5, 6],

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Grammatikopoulos, Ladas and Sficas[8], and Grammatikopoulos, Ladas and Meimaridou[7] studied the asymptotic behavior of the solutions of the following NDDEs:

$$\frac{d}{dt}[y(t)+py(t-\tau)]+qy(t-\sigma)=0, \quad t\geqslant t_0,$$

$$\frac{d}{dt}[y(t)+P(t)y(t-\tau)]+Q(t)y(t-\sigma)=0, \quad t\geqslant t_0,$$

$$\frac{d^2}{dt^2}[y(t)+P(t)y(t-\tau)]+Q(t)y(t-\sigma)=0, \quad t\geqslant t_0,$$

$$\frac{d^n}{dt^n}[y(t)+py(t-\tau)]+qy(t-\sigma)=0, \quad t\geqslant t_0,$$

where $n\geqslant 1$, P, $Q\in([t_0, \infty), \mathbb{R})$ and p, q, τ, and σ are real numbers.

For some results on nonlinear second order NDDEs[16].

In this paper we deal with the asymptotic and oscillatory behavior of the NDDE of order $n\geqslant 1$,

$$\frac{d^n}{dt^n}[y(t)+P(t)y(t-\tau)]+Q(t)y(t-\sigma)=0, \quad t\geqslant t_0, \tag{1}$$

where P, $Q\in C([t_0, \infty), \mathbb{R})$ and the delays τ and σ are nonnegative constants.

In Section 2 we study the asymptotic behavior of the nonoscillatory solutions of Equation (1), in Section 3 we study the oscillation of NDDEs of odd order and in Section 4 we examine the oscillation of even order NDDEs.

Let $\phi\in C([t_0-\rho, t_0], \mathbb{R})$, where $\rho=\max\{\tau, \sigma\}$, be a given function and let z_k, $k=0, 1, \cdots, n-1$ be given constants. Using the method of steps it follows that Equation (1) has a unique solution $y\in C([t_0-\rho, \infty), \mathbb{R})$ in the sense that

$$y(t)=\phi(t) \text{ for } t\in[t_0-\rho, t_0],$$

$$\frac{d^k}{dt^k}[y(t)+P(t)\phi(t-\tau)]_{t=t_0}=z_k, \quad k=0, 1, \cdots, n-1,$$

$y(t)+P(t)y(t-\tau)$ is n-times continuously differentiable on $[t_0, \infty)$, and $y(t)$ satisfies Equation (1) for all $t\geqslant t_0$. For further questions concerning existence and uniqueness of solutions of NDDEs see Driver[3,4], Bellman and Cooke[1], and Hale[9].

As usual, a solution of Eq uation (1) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

In the sequel, for convenience, we will assume that inequalities about values of functions are satisfied eventually for all large t.

§2. Asymptotic Behavior of Nonoscillatory Solutions

In this section we study the asymptotic behavior of the nonoscillatory solutions of the NDDE

$$\frac{d^n}{dt^n}[y(t)+P(t)y(t-\tau)]+Q(t)y(t-\sigma)=0, \quad t\geqslant t_0, \tag{1}$$

where $n\geqslant 1$, $P, Q\in C([t_0, \infty), \mathbb{R})$, and the delays τ and σ are nonnegative constants.

Throughout this paper, unless otherwise specified, we assume that the following hypotheses are satisfied:

(H_1) There exist constants P_1 and P_2 such that

$$P_1\leqslant P(t)\leqslant P_2.$$

(H_2) There exists a positive constant q such that

$$Q(t)\geqslant q>0.$$

Let $y(t)$ be a solution of Equation (1). Set

$$z(t)=y(t)+P(t)y(t-\tau).$$

The following lemma describes some asymptotic properties of the function $z(t)$ when $y(t)$ is a nonoscillatory solution of Equation (1).

Lemma 1. *Assume that the hypotheses* (H_1) *and* (H_2) *are satisfied. Let $y(t)$ be an eventually positive solution of Equation* (1). *Set*

$$z(t)=y(t)+P(t)y(t-\tau). \tag{2}$$

Then the following statements are true:

(i) *For each $i=0, 1, \cdots, n-1$ the function $z^{(i)}(t)$ is strictly monotonic and either*

$$\lim_{t\to\infty} z^{(i)}(t)=-\infty \tag{3}$$

or

$$\lim_{t\to\infty} z^{(i)}(t)=0 \quad and \quad z^{(i)}(t)z^{(i+1)}(t)<0. \tag{4}$$

(ii) *For n even the function $z(t)$ is negative.*

(iii) *Assume that $p_2<-1$ and that n is odd. Then* (3) *holds.*

(iv) *Assume that $p_1\geqslant -1$ and that n is odd or even. Then* (4) *holds, and in particular, $z(t)$ is bounded.*

Proof (i) From Equation (1) we find

$$z^{(n)}(t)=-Q(t)y(t-\sigma)\leqslant -qy(t-\sigma)<0 \tag{5}$$

which implies that $z^{(n-1)}(t)$ is a strictly decreasing function of t, while $z^{(i)}(t)$, $i=0, 1, \cdots, n-2$ are strictly monotonic functions of t. Therefore either

$$\lim_{t\to\infty} z^{(n-1)}(t)=-\infty, \tag{6}$$

or

$$\lim_{t\to\infty} z^{(n-1)}(t)\equiv l \text{ is finite}. \tag{7}$$

Assume that (6) is satisfied. Then it is easily seen that (3) holds.

Assume now that (7) holds. Then integrating both sides of (5) from t_1 to t, with t_1 sufficiently large, and letting $t\to\infty$, we find

$$\int_{t_1}^{\infty} qy(t-\sigma)ds\leqslant z^{(n-1)}(t_1)-l$$

which implies that $y\in L^1[t_1, \infty)$. Thus, in view of (H_1), $z\in L^1[t_1, \infty)$. Since $z(t)$ is monotonic, it follows that

$$\lim_{t\to\infty} z(t)=0 \tag{8}$$

and so also $l=0$. Finally, from (8) we conclude that consecutive derivarives of $z(t)$ alternate sign, that is, for each $i=0, 1, \cdots, n-1$

$$z^{(i)}(t)z^{(i+1)}(t)<0.$$

(ii) Clearly for n even, either (3) or (4) implies that

$$z(t)<0.$$

(iii) If (3) were false, then (4) would hold and so

$$z(t)>0. \tag{9}$$

Therefore

$$y(t)>-P(t)y(t-\tau)\geqslant -P_2y(t-\tau)$$

and by iteration

$$y(t+k\tau)>(-p_2)^k y(t)\to\infty \text{ as } k\to\infty.$$

Hence

$$\lim_{t\to\infty} y(t)=\infty$$

and (5) implies that

$$\lim_{t\to\infty} z^{(n)}(t)=-\infty.$$

Thus

$$\lim_{t\to\infty} z(t)=-\infty,$$

which contradicts (9) and proves (3).

(iv) If (4) were false, then, from (3),

$$\lim_{t\to\infty} z(t)=-\infty \tag{10}$$

and so

$$z(t)<0.$$

Hence

$$y(t)<-P(t)y(t-\tau)\leqslant -p_1y(t-\tau)\leqslant y(t-\tau),$$

which implies that $y(t)$ is a bounded function. This contradicts (10). The proof of the lemma is complete.

Using the asymptotic properties of the function $z(t)$, we obtain the following result about the asymptotic behavior of the nonoscillatory solutions of Equation (1).

Theorem 1. *Consider the NDDE* (1) *and assume that the hypotheses* (H_1) *and* (H_2) *are satisfied. Then the following statements are true:*

(a) *Assume that*

$$n \text{ is odd and } p_2<-1.$$

Then every nonoscillatory solution $y(t)$ *of Equation* (1) *tencs to* $+\infty$ *or* $-\infty$ *as* $t\to\infty$.

(b) *Assume that*

$$n \text{ is odd and } p_1\geqslant 0 \tag{11}$$

or

$$(n \text{ is even or odd and}) \ -1<p_1\leqslant p_2<0. \tag{12}$$

Then every nonoscillatory solution $y(t)$ *of Equation* (1) *tends to zero as* $t\to\infty$.

Proof As the negative of a solution of Equation (1) is also a solution of the same equation, it suffices to prove the theorem for an eventually positive solution $y(t)$ of Equaion (1).

(a) Set

$$z(t)=y(t)+P(t)y(t-\tau).$$

Then from Lemma 1 (iii) we have

$$\lim_{t\to\infty} z(t)=-\infty.$$

Observe that

$$p_1 y(t-\tau)\leqslant P(t)y(t-\tau)<z(t)\to-\infty \text{ as } t\to\infty$$

and so

$$\lim_{t\to\infty} y(t)=\infty.$$

(b) Assume that (11) holds. Then, using Lemma 1 (iv), we find

$$0<y(t)\leqslant z(t)\to 0 \text{ as } t\to\infty$$

and so

$$\lim_{t\to\infty} y(t)=0.$$

Next assume that (12) is satisfied. Then Lemma 1 (iv) implies that (4) holds. Depending on whether n is even or odd, we distinguish the following two cases:

Case 1. n is even. In this case Lemma 1 (ii) implies that

$$z(t)<0$$

and hence

$$y(t)<-P(t)y(t-\tau)<y(t-\tau).$$

Therefore, $y(t)$ is a bounded function. Assume, for the sake of contradiction, that

$$\limsup_{t\to\infty} y(t)\equiv s>0.$$

Let $\{t_k\}$ be a sequence of points such that

$$\lim_{k\to\infty} t_k=\infty \quad\text{and}\quad \lim_{k\to\infty} y(t_k)=s.$$

Then, for sufficiently large k,

$$z(t_k)=y(t_k)+P(t_k)y(t_k-\tau)\geqslant y(t_k)+p_1 y(t_k-\tau)$$

and hence

$$\limsup_{k\to\infty} y(t_k-\tau)\geqslant\frac{s}{-p_1}>s,$$

which is a contradiction.

Case 2. n is odd. First we will prove that $y(t)$ is a bounded function. To this end, observe that from Lemma 1 (iv) we have

$$z(t)>0,\ z'(t)<0 \text{ and } \lim_{t\to\infty} z(t)=0.$$

Therefore, there is a positive constant B such that

$$z(t)\leqslant B$$

and so

$$y(t)<-P(t)y(t-\tau)+B\leqslant -p_1 y(t-\tau)+B. \tag{13}$$

Assume, for the sake of contradiction, that $y(t)$ is not bounded. Then there is a sequence of points $\{t_k\}$ such that

$$\lim_{k\to\infty} t_k=\infty, \quad \lim_{k\to\infty} y(t_k)=\infty, \quad \text{and} \quad y(t_k)=\max_{t_0\le s\le t_k} y(s).$$

Thus, from (13) we have

$$y(t_k)<-p_1y(t_k-\tau)+B\leqslant -p_1y(t_k)+B$$

or equivalently

$$(1+p_1)y(t_k)\leqslant B,$$

which as $k\to\infty$ leads to a contradiction. Set

$$s\equiv\limsup_{t\to\infty} y(t)$$

which exists, because $y(t)$ is bounded. Let $\{t_k\}$ be a sequence of points such that

$$\lim_{k\to\infty} t_k=\infty \quad \text{and} \quad \lim_{k\to\infty} y(t_k)=s.$$

Then, for sufficiently large k,

$$z(t_k)=y(t_k)+P(t)y(t_k)\geqslant y(t_k)+p_1y(t_k-\tau),$$

which as $k\to\infty$ implies that $s=0$. The proof of the theorem is complete.

The following examples show that Theorem 1 may not be true, if either the hypothesis (H_1) or (H_2) is not satisfied.

Example 1. In the NDDE

$$\frac{d^n}{dt^n}[y(t)-e^t y(t-1)]+e^{-2}y(t-2)=0,\ t\geqslant 1,\ n \text{ is odd},$$

the hypothesis (H_1) of Theorem 1 (a) is not satisfied. Note that $y(t)=e^{-1}$ is a solution of this equation with $\lim_{t\to\infty} y(t)=0$.

Example 2. In the NDDE

$$\frac{d^2}{dt^2}\Big[y(t)+\Big[-\frac{1}{2}+(t-1)^{-1/2}\Big]y(t-1)\Big]$$
$$+\frac{1}{4}(t-2)^{-1/2}\Big[t^{-3/2}-\frac{1}{2}(t-1)^{-3/2}\Big]y(t-2)=0,\quad t\geqslant 3,$$

the hypothesis (H_2) of Theorem 1 (b) is not satisfied. Note that $y(t)=t^{1/2}$ is a solution of this equation with $\lim_{t\to\infty} y(t)=\infty$.

§ 3. Sufficient Conditions for Oscillation of Odd Order NDDEs

In this section we study the oscillatory properties of the solutions of the odd order NDDE

$$\frac{d^n}{dt^n}[y(t)+P(t)y(t-\tau)]+Q(t)y(t-\sigma)=0,\quad t\geqslant t_0, \tag{1}$$

where $P, Q\in C([t_0, \infty), \mathbb{R})$ and the delays τ and σ are nonnegative constants.

The following lemma (as well as Lemma 3 of the next section), which will be used in the proofs of Theorems 2, 3, and 6, has been extracted from results due to

Ladas and Stavroulakis[12,13].

Lemma 2. *Assu me that n is odd and r and μ are positive constants such that*

$$r^{1/n}\frac{\mu}{n}>\frac{1}{e}.$$

Then, the following statements are true:

(i) *the inequality*

$$x^{(n)}(t)-rx(t+\mu)\leqslant 0$$

has no eventually negative solution;

(ii) *the inequality*

$$x^{(n)}(t)-rx(t+\mu)\geqslant 0$$

has no eventually positive solution;

(iii) *the inequality*

$$x^{(n)}(t)+rx(t-\mu)\geqslant 0$$

has no eventually negative solution;

(iv) *the inequality*

$$x^{(n)}(t)+rx(t-\mu)\leqslant 0$$

has no eventually positive solution.

Let $y(t)$ be a solution of Equation (1). Set

$$z(t)=y(t)+P(t)y(t-\tau).$$

Then a direct substitution shows that $z(t)$ is an n times continuously differentiable solution of the NDDE

$$z^{(n)}(t)+R(t)z^{(n)}(t-\tau)+Q(t)z(t-\sigma)=0,\ t\geqslant t_0, \tag{14}$$

where

$$R(t)=P(t-\sigma)\frac{Q(t)}{Q(t-\tau)}.$$

Theorems 2, 3, and 6 below provide sufficient conditions for the oscillation of all solutions, while Theorems 4 and 5 deal with unbounded solutions only.

Theorem 2. *Consider the NDDE* (1) *and assume that n is odd and that the hypotheses* (H_1) *and* (H_2) *are satisfied with*

$$p_2<-1.$$

Suppose also that there exists a positive constant r such that $\tau>\sigma$,

$$\frac{Q(t)}{P(t+\tau-\sigma)}\leqslant -r \tag{15}$$

and

$$r^{1/n}\frac{\tau-\sigma}{n}>\frac{1}{e}. \tag{16}$$

Then every solution of Equation (1) *oscillates.*

Proof Otherwise there is an eventually positive solution $y(t)$ of Equation (1). Set

$$z(t)=y(t)+P(t)y(t-\tau).$$

Then

$$z^{(n)}(t) = -Q(t)y(t-\sigma) \leqslant 0$$

and also, Lemma 1 (iii) implies that

$$z(t) < 0.$$

As $z^{(n)}(t) \leqslant 0$, from Equation (14) it follows that

$$z^{(n)}(t) + \frac{Q(t)}{P(t+\tau-\sigma)} z[t+(\tau-\sigma)[\leqslant 0$$

which, by Lemma 2 (i), (15) and (16), has no eventually negative solution. This is a contradiction. The proof is complete.

Theorem 3. *Consider the NDDE* (1) *and assume that n is odd and that the hypotheses* (H_1) *and* (H_2) *are satisfied with*

$$-1 \leqslant p_1 \leqslant p_2 \leqslant 0.$$

Suppose also that

$$q^{1/n} \frac{\sigma}{n} > \frac{1}{e}. \tag{17}$$

Then every solution of Equation (1) *oscillates.*

Proof Otherwise there is an eventually positive solution $y(t)$ of Equation (1). Set

$$z(t) = y(t) + P(t)y(t-\tau).$$

Then $z(t)$ is a solution of Equation (14). Also

$$z^{(n)}(t) < 0,$$

and from Lemma 1 (iv)

$$z(t) > 0. \tag{18}$$

Then, from Equation (14) we find

$$z^{(n)}(t) + qz(t-\sigma) < 0.$$

But, because of Lemma 2 (iv) and (17), it is impossible for this inequality to have an eventually positive solution. This contradicts (18). The proof of the theorem is complete.

In Theorems 4 and 5 below the hypothesis (H_2) is not required.

Theorem 4. *Consider the NDDE* (1) *and assume that n is odd,*

$$-1 < p_1 \leqslant P(t) \leqslant 0,$$

$$Q(t) \geqslant 0,$$

and

$$\int_{t_0}^{\infty} Q(s)ds = \infty. \tag{19}$$

Then every unbounded solution of Equation (1) *oscillates.*

Proof Otherwise Equation (1) has an eventually positive unbounded solution $y(t)$. Set

$$z(t) = y(t) + P(t)y(t-\tau).$$

Then

$$z^{(n)}(t)=-Q(t)y(t-\sigma)\leqslant 0. \tag{20}$$

We will prove that

$$z(t)>0 \tag{21}$$

and

$$z'(t)>0. \tag{22}$$

If (21) were false then $z(t)\leqslant 0$, which implies that

$$y(t)\leqslant -P(t)y(t-\tau)\leqslant y(t-\tau),$$

that is, $y(t)$ is bounded. On the other hand, if (22) were false then $z'(t)\leqslant 0$ and in particular $z(z)$ would be a bounded function. Thus, there would exist a positive constant B such that

$$y(t)\leqslant -P(t)y(t-\tau)+B\leqslant -p_1y(t-\tau)+B. \tag{23}$$

As $y(t)$ is unbounded, there must exist a sequence of points $\{t_k\}$ such that

$$\lim_{k\to\infty} t_k=\infty, \quad \lim_{k\to\infty} y(t_k)=\infty, \quad \text{and} \quad y(t_k)=\max_{t_0\leqslant s\leqslant t_k} y(s).$$

Then, from (23)

$$y(t_k)\leqslant -p_1y(t_k)+B,$$

or equivalently

$$(1+p_1)y(t_k)\leqslant B,$$

which contradicts the fact that $y(t_k)$ is unbounded. Thus (21) and (22) have been established. Since

$$0<z(t)<y(t),$$

Equation (20) implies that

$$z^{(n)}(t)+Q(t)z(t-\sigma)\leqslant 0.$$

Integrating from t_1 to t, for t_1 sufficiently large, we find

$$z^{(n-1)}(t)-z^{(n-1)}(t_1)+z(t_1-\sigma)\int_{t_1}^{t} Q(s)ds\leqslant 0,$$

which, in view of (19), implies that

$$z^{(n-1)}(t)<0. \tag{24}$$

From (24) and (20) it follows that $z(t)<0$, which contradicts (21). The proof is complete.

Theorem 5. *Consider the NDDE* (1) *and assume that n is odd,*

$$0\leqslant P(t)\leqslant 1,\ Q(t)\geqslant 0,$$

and

$$\int_{t_0}^{\infty} Q(s)[1-P(s-\sigma)]ds=\infty.$$

Then every unbounded solution of Equation (1) *oscillates.*

Proof Otherwise, there is an eventually positive unbounded solution $y(t)$ of Fquation (1). Set

$$z(t)=y(t)+P(t)y(t-\tau).$$

Then

$$z^{(n)}(t)=-Q(t)y(t-\sigma)\leqslant 0. \tag{25}$$

Clearly

$$z(t)>0,\ z(t)>y(t),\ \text{and } z(t) \text{ is unbounded.}$$

This is already impossible for $n=1$, while for $n\geqslant 3$ we have $z'(t)>0$ and so

$$z(t-\sigma)<y(t-\sigma)+P(t-\sigma)z(t-\tau-\sigma)<y(t-\sigma)+P(t-\sigma)z(t-\sigma)$$

or

$$y(t-\sigma)>[1-P(t-\sigma)]z(t-\sigma).$$

Thus, Equation (25) yields

$$z^{(n)}(t)+Q(t)[1-P(t-\sigma)]z(t-\sigma)\leqslant 0.$$

Integrating from t_1 to t, for t_1 sufficiently large, we find

$$z^{(n-1)}(t)-z^{(n-1)}(t_1)+z(t_1-\sigma)\int_{t_1}^{t}Q(s)[1-P(t-\sigma)]ds\leqslant 0,$$

which as $t\to\infty$ leads to a contradiction. The proof is complete.

Remark. For $n=1$, the conclusion of Theorem 5 remains true under the hypotheses

$$P(t)\geqslant p_1>-1 \quad\text{and}\quad Q(t)\geqslant 0$$

only. The proof of this follows from Theorem 3(i) in [8].

Theorem 6. *Consider the NDDE* (1) *and assume that* n *is odd,* (H_2) *is satisfied,* $Q(t)$ *is a* τ*-periodic function and*

$$P(t)\equiv p\in\mathbb{R}.$$

Then each of the following conditions implies that every solution of Equation (1) *oscillates*:

$$\text{(i) } p<-1,\ \tau>\sigma \text{ and } \left(-\frac{q}{1+p}\right)^{1/n}\frac{\tau-\sigma}{n}>\frac{1}{e}; \tag{26}$$

$$\text{(ii) } p=-1; \tag{27}$$

$$\text{(iii) } p>-1,\ \sigma>\tau \text{ and } \left(\frac{q}{1+p}\right)^{1/n}\frac{\sigma-\tau}{n}>\frac{1}{e}. \tag{28}$$

Proof Assume that one of the conditions (i)−(iii) is satisfied and that, contrary to the conclusion of the theorem, Equation (1) has an eventually positive solution $y(t)$. Set

$$z(t)=y(t)+py(t-\tau) \quad\text{and}\quad w(t)=z(t)+pz(t-\tau).$$

Then

$$z^{(n)}(t)=-Q(t)y(t-\sigma)\leqslant 0,$$
$$w^{(n)}(t)=-Q(t)z(t-\sigma),$$

and

$$w^{(n)}(t)+pw^{(n)}(t-\tau)+Q(t)w(t-\sigma)=0. \tag{29}$$

First, assume that (26) is satisfied. Then by Lemma 1 (iii) it follows that (3) is satisfied. This implies that

$$w^{(n)}(t)=-Q(t)z(t-\sigma)\geqslant -qz(t-\sigma)\to+\infty$$

and so

$$w(t)>0. \tag{30}$$

Also

$$w^{(n)}(t-\tau)=-Q(t)z(t-\tau-\sigma)\leqslant -Q(t)z(t-\sigma)=w^{(n)}(t). \tag{31}$$

Substituting (31) into Equation (29) we find

$$w^{(n)}(t)+\frac{q}{1+p}w(t+\tau-\sigma)\geqslant 0. \tag{32}$$

But, in view of Lemma 2 (ii) and (26), Inequality (32) cannot have an eventually positive solution. This contradicts (30) and proves the theorem when (26) is satisfied.

Next, assume that (27) holds. Then, by Lemma 1 (iv) it follows that (4) is satisfied. This implies that (31) is true and that

$$w(t)>0.$$

Hence (29) yields

$$Q(t)w(t-\sigma)\leqslant 0,$$

which contradicts the fact that $w(t)$ is positive.

Finally, assume that (28) is satisfied. Then, again, (31) holds and (29) implies that

$$w^{(n)}(t)+\frac{q}{1+p}w(t-(\sigma-\tau))\leqslant 0. \tag{33}$$

But, in view of Lemma 2 (iv) and (28), Inequality (33) cannot have an eventually positive solution. This contradicts (30). The proof of the theorem is complete.

§ 4. Sufficient Conditions for Oscillation of Even Order NDDEs

In this section we study the oscillatory properties of the solutions of the even order NDDE

$$\frac{d^n}{dt^n}[y(t)+P(t)y(t-\tau)]+Q(t)y(t-\sigma)=0, \quad t\geqslant t_0, \tag{1}$$

where P, $Q\in C([t_0, \infty), \mathbb{R})$ and the delays τ and σ are nonnegative constsnta.

The following lemma, which will be used in the proofs of Theorems 8 an 11 below, has been extracted from results due to Ladas and Stavroulakis[12].

Lemma 3. *Assume that n is even and r and μ are positive constants such that*

$$r^{1/n}\frac{\mu}{n}>\frac{1}{e}.$$

Then the inequality

$$x^{(n)}(t)-rx(t-\mu)\leqslant 0$$

has no eventually negative bounded solution.

Theorems 7—10 below provide sufficient conditions for the oscillation of all solutions of Equation (1), while Theorems 11 and 12 deal with the oscillation of all bounded and all unbounded solutions respectively.

Theorem 7. *Consider the NDDE* (1) *and assume that n is even and that the* *hypotheses* (H_1) *and* (H_2) *are satisfied. Furthermore assume that* $P(t)$ *is not eventually negative. Then every solution of Equation* (1) *oscillates.*

Proof Assume, for the sake of contradiction, that $y(t)$ is an eventually positive solution of Equation (1). Set

$$z(t)=y(t)+P(t)y(t-\tau). \tag{2}$$

Then, eventually, $z(t)$ takes nonnegative values. But, since n is even, Lemma 1 (ii) implies that $z(t)$ is eventually negative. This contradiction completes the proof.

The example below illustrates Theorem 7.

Example 3. The NDDE

$$\frac{d^2}{dt^2}\left[y(t)+\left(\frac{1}{2}+\cos t\right)y(t-2\pi)\right]+\left(\frac{3}{2}+\cos t\right)y(t-4\pi)=0, \quad t\geqslant 0,$$

satisfies the hypotheses of Theorem 7. Therefore, every solution of this equation oscillates. For example, $y(t)=\dfrac{\cos t}{3/2+\cos t}$ is an oscillatory solution.

The following example shows that if we remove the hypothesis (H_2) from Theorem 7, the result may not be true.

Example 4. The NDDE

$$\frac{d^2}{dt^2}[y(t)+(t-1)^{-1/2}y(t-1)]+1/4t^{-3/3}(t-2)^{-1/2}y(t-2)=0, \quad t>2,$$

satisfies all the hypotheses of Theorem 7 except (H_2). Note that $y(t)=t^{1/2}$ is a nonoscillatory solution of this equation.

Theorem 8. *Consider the NDDE* (1) *and assume that* n *is even and that the hypotheses* (H_1) *and* (H_2) *are satisfied with*

$$-1\leqslant p_1\leqslant p_2<0.$$

Suppose also that there is a positive constant r *such that* $\sigma>\tau$,

$$\frac{Q(t)}{P(t+\tau-\sigma)}\leqslant -r \tag{34}$$

and

$$r^{1/n}\frac{\sigma-\tau}{n}>\frac{1}{e}. \tag{35}$$

Then every solution of Equation (1) *oscillates.*

Proof Otherwise there is an eventually positive solution $y(t)$ of Equation (1). Set

$$z(t)=y(t)+P(t)y(t-\tau).$$

Then $z(t)$ is a solution of Equation (14). Clearly

$$z^{(n)}(t)<0 \tag{36}$$

and from Lemma 1 (ii) and (iv) we conclude that $z(t)$ is an eventually negative and bounded function. Using (36), from Equation (14) we obtain

$$R(t)z^{(n)}(t-\tau)+Q(t)z(t-\sigma)>0.$$

Hence

$$z^{(n)}(t)+\frac{Q(t)}{P(t-\tau-\sigma)}z(t-(\sigma-\tau))<0,$$

which, in view of (34), leads to the inequality

$$z^{(n)}(t)-rz(t-(\sigma-\tau))<0.$$

But, because of (35), Lemma 3 implies that it is impossible for this inequality to have an eventually negative bounded solution. This completes the proof of the theorem.

Example 5. For the NDDE

$$\frac{d^2}{dt^2}[y(t)-(4+e^{-t})y(t-1)]+e(4-e)y(t-2)=0,\ t\geqslant 1,$$

the hypothesis that $-1\leqslant p_1$ is not satisfied. Note that $y(t)=e^t$ is a nonoscillatory solution of this equation.

In Theorems 9 and 10 the hypothesis (H_2) is not required.

Theorem 9. *Consider the NDDE* (1) *and assume that* n *is even. Suppose also that*

$$Q(t)\geqslant 0,\ Q(t)\not\equiv 0 \text{ and } \tau\text{-periodic},$$

$$0\leqslant P(t)\equiv p \text{ is constant.}$$

Then every solution of Equation (1) *oscillates.*

Proof Otherwise there is an eventually positive solution $y(t)$ of Equation (1). Set

$$z(t)=y(t)+py(t-\tau) \quad \text{and} \quad w(t)=z(t)+pz(t-\tau).$$

Then

$$z(t)>0 \text{ and } w(t)>0.$$

Also

$$z^{(n)}(t)=-Q(t)y(t-\sigma)\leqslant 0,$$

and

$$w^{(n)}(t)=-Q(t)z(t-\sigma)\leqslant 0.$$

We claim that

$$z^{(n-1)}(t)\geqslant 0 \text{ and } w^{(n-1)}(t)\geqslant 0. \tag{37}$$

Otherwise

$$z^{(n-1)}(t)<0 \text{ or } w^{(n-1)}(t)<0,$$

which together with $z^{(n)}(t)\leqslant 0$ and $w^{(n)}(t)\leqslant 0$ implies that

$$z(t)<0 \text{ or } w(t)<0,$$

which is a contradiction.

Next, we claim that

$$z'(t)\geqslant 0 \text{ and } w'(t)\geqslant 0.$$

Otherwise

$$z'(t)<0 \text{ or } w'(t)<0.$$

But any one of these inequalities implies that the higher derivatives of odd order of that function are also negative. This contradicts (37). Thus we have proved that

$z(t)$ and $w(t)$ are increasing functions of t. Observe now that $w(t)$ is a continuously differentiable solution of the NDDE

$$w^{(n)}(t)+pw^{(n)}(t-\tau)+Q(t)w(t-\sigma)=0. \tag{38}$$

As

$$w^{(n)}(t-\tau)=-Q(t)z(t-\sigma-\tau)\geqslant -Q(t)z(t-\sigma)=w^{(n)}(t),$$

Equation (38) implies that

$$w^{(n)}(t)+\frac{1}{1+p}Q(t)w(t-\sigma)\leqslant 0.$$

Integrating both sides of this inequality from t_1 to t, with t_1 sufficiently large, we find that

$$w^{(n-1)}(t)-w^{(n-1)}(t_1)+\frac{1}{1+p}w(t_1-\sigma)\int_{t_1}^{\infty}Q(s)ds\leqslant 0,$$

which leads to a contradiction as $t\to\infty$. The proof is complete.

Theorem 10. *Consider the NDDE* (1) *and assume that n is even. Assume also that*

$$0\leqslant P(t)\leqslant 1,\ Q(t)\geqslant 0,$$

and that

$$\int_{t_1}^{\infty}Q(s)[1-P(t-\sigma)]ds=\infty.$$

Then every solution of Equation (1) *oscillates.*

Proof Assume, for the sake of contradiction, that $y(t)$ is an eventually positive solution of Equation (1). Set

$$z(t)=y(t)+P(t)y(t-\tau). \tag{39}$$

Then

$$z(t)\geqslant 0 \tag{40}$$

and

$$z^{(n)}(t)\leqslant 0. \tag{41}$$

Hence $z^{(n-1)}(t)$ is a decreasing function of t. We claim that

$$z^{(n-1)}(t)\geqslant 0. \tag{42}$$

Otherwise

$$z^{(n-1)}(t)<0,$$

which together with (41) implies that

$$\lim_{t\to\infty}z^{(k)}(t)=-\infty,\ k=0,\ 1,\ \cdots,\ n-2.$$

But this contradicts (40).

Next, observe that from Equation (1) we have

$$z^{(n)}(t)+Q(t)y(t-\sigma)=0. \tag{43}$$

Using (39) in (43), we see that

$$z^{(n)}(t)+Q(t)[z(t-\sigma)-P(t-\sigma)y(t-\tau-\sigma)]=0. \tag{44}$$

As

$$z(t)>y(t),$$

(44) yields

$$z^{(n)}(t)+Q(t)[z(t-\sigma)-P(t-\sigma)z(t-\tau-\sigma)]\leqslant 0,$$

which, in view of (42), leads to the inequality

$$z^{(n)}(t)+Q(t)[1-P(t-\sigma)]z(t-\sigma)\leqslant 0. \tag{45}$$

Integrating both sides of (45) from t_1 to t, with t_1 sufficiently large, we find that

$$z^{(n-1)}(t)-z^{(n-1)}(t_1)+z(t_1-\sigma)\int_{t_1}^{t}Q(s)[1-P(s-\sigma)]ds\leqslant 0,$$

which, as $t\to\infty$, leads to a contradiction. The proof is complete.

Theorem 11. *Consider the NDDE* (1) *and assume that n is even. Assume also that the hypotheses* (H_1) *and* (H_2) *are satisfied with*

$$p_2<0$$

and that there is a positive number r such that $\sigma>\tau$,

$$\frac{Q(t)}{P(t+\tau-\sigma)}\leqslant -r \tag{46}$$

and

$$r^{1/n}\frac{\sigma-\tau}{n}>\frac{1}{e}.$$

Then every bounded solution of Equation (1) *oscillates.*

Proof Otherwise there is an eventually positive bounded solution $y(t)$ of Equation (1). Set

$$z(t)=y(t)+P(t)y(t-\tau).$$

Then

$$z^{(n)}(t)<0. \tag{47}$$

Since n is even,

$$z(t)<0. \tag{48}$$

Therefore, $z(t)$ is an eventually negative and bounded solution of Equation (1). In view of (48), (47) and (46), we obtain

$$z^{(n)}(t)-rz(t-(\sigma-\tau))<0.$$

But, Lemma 3 implies that the above inequality has no eventually negative bounded solution. This contradicts (48) and the proof is complete.

Example 5, which we presented earlier, also illustrates that under the hypothses of Theorem 11, Equation (1) may have unbounded nonoscillatory solutions.

In the next result the hypothesis (H_2) is not required.

Theorem 12. *Consider the NDDE* (1). *Assume that n is even,*

$$-1\leqslant P(t)\leqslant 0,$$

$$Q(t)\geqslant 0,$$

and that

$$\int_{t_0}^{\infty}Q(s)ds=\infty.$$

Then every unbounded solution of Equation (1) *oscillates.*

Proof Assume, for the sake of contradiction, that $y(t)$ is an unbounded positive solution of Equation (1). Set

$$z(t)=y(t)+P(t)y(t-\tau).$$

We have

$$z^{(n)}(t) = -Q(t)y(t-\sigma) \leqslant 0$$

and so $z^{(i)}(t)$, for $i=0, 1, \cdots, n-1$, are monotonic functions. We claim that

$$z^{(n-1)}(t) \geqslant 0,\ z'(t) \geqslant 0, \text{ and } z(t) \geqslant 0.$$

Otherwise $z(t) < 0$ and so

$$y(t) < -P(t)y(t-\tau) \leqslant y(t-\tau),$$

which is impossible because $y(t)$ is unbounded. Integrating (1) from t_1 to t, with sufficiently large t_1, we find

$$z^{(n-1)}(t) - z^{(n-1)}(t_1) + z(t_1-\sigma)\int_{t_1}^{t} Q(s)ds \leqslant 0,$$

which, as $t \to \infty$, is impossible. The proof is complete.

Example 6. The NDDE

$$\frac{d^2}{dt^2}\left[y(t) - \frac{2e^{\pi/2}+e^{-\pi/2}}{e^{3\pi/2}+2e^{-3\pi/2}}\, y(t-2\pi)\right] + \frac{2(e^{2\pi}-e^{-2\pi})}{e^{3\pi/2}+2e^{-3\pi/2}}\, y\left(t-\frac{\pi}{2}\right) = 0, \quad t \geqslant 0,$$

satisfies the hypotheses of Theorem 12. Therefore, every unbounded solution of this equation oscillates. For example, $y(t) = e^t \cos t$ is such a solution. On the other hand, the bounded solutions of this equation do not have to oscillate. For example, $y(t) = e^{-t}$ is such a solution.

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