Chin. Ann. of Math.
9B (3) 1988

SOME PROPERTIES OF INTERPOLATING BASIC FUNCTIONS IN THE UNIT DISK

Shen Xiechang (沈燮昌)*

Abstract

Consider the sequence of interpolating basic functions. If the sequence is incomplete in the weighted H^p spaces, the characteristic properties of its closure is obtained. Furthermore, if the interpolating points are uniformly separated, then the sequene is a basics in its closure in H^p spaces for $p>1$, and if $0<p<1$, then generally it is not a basis in its closure.

Let $\{a_k\}$, $k=1, 2, \cdots$, be a given sequence in the unit disk the elements of which can coincide with each other. We denote by s_k the number of appearance of a_k in $\{a_1, a_2, \cdots, a_k\}$ and by p_k the number of appearance of a_k in $\{a_j\}$. It is obviously that $1\leqslant s_k\leqslant p_k$, $k=1, 2, \cdots$.

Hereafter we suppose

$$\sum_{k=1}^{+\infty}(1-|a_k|)<+\infty. \tag{1}$$

Under the condition (1) we construct the Blaschke product[1]

$$B(z)=\prod_{k=1}^{+\infty}\frac{a_k-z}{1-\bar{a}_k z}\frac{|a_k|}{a_k}, \tag{2}$$

where $\frac{|a_k|}{a_k}=-1$, when $a_k=0$. It is known[1] that $B(z)$ is analytic in $|z|<1$, $|B(z)|<1$ in $|z|<1$ and $|B(e^{i\theta})|=1$ almost everywhere.

Consider the functions[2]

$$\Omega_k(z)=\frac{(z-a_k)^{s_k-1}}{(s_k-1)!}\frac{B(z)}{(z-a_k)^{p_k}}\sum_{\nu=0}^{p_k-s_k}\alpha_\nu(a_k)(z-a_k)^\nu, \quad k=1, 2, \cdots, \tag{3}$$

where $\alpha_\nu(a_k)$ are the coefficients of Taylor expansion of function $(z-a_k)^{p_k}/B(z)$ at $z=a_k$. It is known that

$$\Omega_k^{(s_j-1)}(a_\nu)=\begin{cases}1, & \text{for } \nu=k,\ s_j=s_\nu,\\ 0, & \text{for } \nu=k,\ s_j\neq s_\nu,\ s_j=1, 2, \cdots, p_\nu,\\ 0, & \text{for } \nu\neq k,\ s_j=1, 2, \cdots, p_\nu.\end{cases} \tag{4}$$

They are Hermite interpolating basic functions with its nodes at $\{a_k\}$ in $|z|<1$. Specially when the elements of $\{a_k\}$ are different from each other, we have

---

Manuscript received January 2, 1986. Revised December 27, 1986.

* Department of Mathematics, Beijing University, Beijing, China.

$$\Omega_k(z)=\frac{B(z)}{B'(a_k)(z-a_k)}, \quad k=1, 2, \cdots, \tag{5}$$

which satisfy

$$\Omega_k(a_\nu)=\begin{cases}1, & \text{for } \nu=k,\\ 0, & \text{for } \nu\neq k.\end{cases}$$

Hence they are Lagrange interpolating basic functions with its nodes at $\{a_k\}$ in $|z|<1$.

Assume that $\sigma(t)$ is a monotone increasing function with the bounded variation and satisfies the following condition

$$\int_0^{2\pi} \ln \sigma'(t)dt>-\infty. \tag{6}$$

We denote by $L^p(\sigma(t); |z|=1)$ the collection of all functions $f(z)$ measurable on $|z|=1$ and satisfying

$$\int_0^{2\pi} |f(e^{it})|^p d\sigma(t)<+\infty, \quad p>0, \tag{7}$$

and by $L^p(\sigma(t); \{a_k\})$, $p>0$, the collection of all functions $f(z)$ belonging to class $L^p(\sigma(t); |z|=1)$ and satisfying the following three properties:

1° There exist two functions $F_1(z)$ and $F_2(z)$ analytic in $|z|<1$ and $|z|>1$ $\left(\text{except at } \frac{1}{a_k}, k=1, 2, \cdots\right)$ recpectively, $F_2(\infty)=0$ and

$$F_1(e^{it})=f(e^{it})=F_2(e^{it}) \tag{8}$$

hold almost everywhere;

2° $F_1(z)w_p^+(z)\in H^p(|z|<1)$; (9)

3° $F_2(z)=\psi(z)B(z)w_p^-(z), \ |z|>1,$ (10)

where $B(z)$ is the Blaschke product defined by (2), $\psi(z)\in H^p(|z|>1)$, $\psi(\infty)=0$, $w_p^+(z)$ and $w_p^-(z)$ are defined by

$$w_p(z)=\exp\left\{\frac{1}{2\pi p}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\ln \sigma'(t)dt\right\} \tag{11}$$

in $|z|<1$ and $|z|>1$ correspondently. It is known[1] that $w_p^+(z)\in D(|z|<1)$, $w_p^-(z)\in D(|z|>1)$ and

$$|w_p^+(e^{it})|=(|w_p^-(e^{it})|)^{-1}=\sigma'(t)^{1/p}, \quad p>0. \tag{12}$$

Theorem 1. *Let $f(z)\in L^p(\sigma(t); |z|=1)$, $p>0$, and $f(z)$ can be approximated by the linear combination of the system of function $\{\Omega_k(z)\}$ in the space $L^p(\sigma(t); |z|=1)$. Then $f(z)\in L^p(\sigma(t); \{a_k\})$.*

Proof Suppose the sequence

$$P_n(z)=\sum_{k=1}^n a_k^{(n)}\Omega_k(z)=B(z)\psi_n(z) \tag{13}$$

satisfies the condition

$$\lim_{n\to+\infty}\int_0^{2\pi} |f(e^{it})-B(e^{it})\psi_n(e^{it})|^p d\sigma(t)=0, \tag{14}$$

where

$$\psi_n(z)=\sum_{k=1}^{n}\frac{a_k^{(n)}}{(s_k-1)!}\sum_{\nu=0}^{p_k-s_k}\alpha_\nu(a_k)(z-a_k)^{s_k-p_k-\nu-1}. \quad (15)$$

Obviously

$$\psi_m(z)\in H^p(|z|>1),\quad \psi_n(\infty)=0,$$
$$B(z)\psi_n(z)\in H^p(|z|<1),\quad n=1,2,\cdots.$$

From (14) and (2) it follows that

1° For any given $\varepsilon>0$, there exists an integer N such that when $n>N$, for every integer m, we have

$$\int_0^{2\pi}|B(e^{it})\psi_n(e^{it})-B(e^{it})\psi_{n+m}(e^{it})|^p\sigma'(t)dt<\varepsilon; \quad (16)$$

2° There exists a constant C^* such that for any integer n we have

$$\int_0^{2\pi}|B(e^{it})\psi_n(e^{it})|^p d\sigma(t)\leqslant C. \quad (17)$$

Now we are going to prove the first part of the 1° and 2° in the definition of $L^p(\sigma(t);\{a_k\})$.

At first from the condition (6) and inequality (17) we deduce

$$\int_0^{2\pi}\ln^+|B(e^{it})\psi_n(e^{it})|dt\leqslant\int_0^{2\pi}\ln^+|B(e^{it})\psi_n(e^{it})|d\sigma(t)+\int_0^{2\pi}\ln^+(1/\sigma'(t))dt$$
$$\leqslant\frac{1}{p}\int_0^{2\pi}|B(e^{it})\psi_n(e^{it})|^p d\sigma(t)+\int_0^{2\pi}\ln^+(1/\sigma'(t))dt\leqslant C. \quad (18)$$

Hence from the properties of subharmonic functions and (15) it follows that for every r, $0<r<1$,

$$\ln^+|B(re^{i\varphi})\psi_n(re^{i\varphi})|\leqslant\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(t-\varphi)+r^2}\ln^+|B(e^{it})\psi_n(e^{it})|dt$$
$$\leqslant\frac{1+r}{1-r}C. \quad (19)$$

It means that the system of functions $\{B(z)\psi_n(z)\}$ analytic in $|z|<1$ is uniformly bounded in the interior of $|z|<1$. Thus using the theory of normal family we deduce that there exists a subsequence of $\{B(z)\psi_n(z)\}$ (without loss of generality we assume it is just $\{B(z)\psi_n(z)\}$, which converges uniformly in the interior of $|z|<1$ to some analytic function $F_1(z)$:

$$\lim_{n\to+\infty}B(z)\psi_n(z)=F_1(z),\quad |z|<1, \quad (20)$$

and satisfies

$$\int_0^{2\pi}\ln^+|F_1(re^{it})|dt=\lim_{n\to+\infty}\int_0^{2\pi}\ln^+|B(re^{it})\psi_n(re^{it})|dt$$
$$\leqslant\varlimsup_{n\to+\infty}\int_0^{2\pi}\ln^+|B(e^{it})\psi_n(e^{it})|dt\leqslant C,$$

where the last inequality is obtained by using (18). Hence $F_1(z)\in A(|z|<1)$ (see the definitions in [1]). Thus it has angular boundary values on $|z|=1$ almost

* Here and after we denoted by C constant, which may take different values.

everywhere.

Besides, from (12), (16) and (17) we arrive at

$$\int_0^{2\pi} |B(e^{it})\psi_n(e^{it})w_p^+(e^{it})-B(e^{it})\psi_{n+m}(e^{it})w_p^+(e^{it})|^p dt<\varepsilon \tag{21}$$

and

$$\int_0^{2\pi} |B(e^{it})\psi_n(e^{it})w_p^+(e^{it})|^p dt\leqslant C. \tag{22}$$

Obiviously $B(z)\psi_n(z)\in H^p(|z|<1)$. Consequently $B(z)\psi_n(z)\in D(|z|<1)$. Since $w_p^+(z)\in D(|z|<1)$, using the property that class $D(|z|<1)$ is a ring[1], we have

$$B(z)\psi_n(z)w_p^+(z)\in D(|z|<1).$$

Using (22) from the boundary properties of analytic functions we know

$$B(z)\psi_n(z)w_p^+(z)\in H^p(|z|<1).$$

By using the properties of functions in H^p space we know that for each r, $0<r<1$,

$$\int_0^{2\pi} |B(re^{it})\psi_n(re^{it})w_p^+(re^{it})-B(re^{it})\psi_{n+m}(re^{it})w_p^+(re^{it})|^p dt$$
$$\leqslant\int_0^{2\pi} |B(e^{it})\psi_n(e^{it})w_p^+(e^{it})-B(e^{it})\psi_{n+m}(e^{it})w_p^+(e^{it})|^p dt<\varepsilon.$$

Let $m\to+\infty$. Using (20) from above inequality it follows that

$$\int_0^{2\pi} |B(re^{it})\psi_n(re^{it})w_p^+(re^{it})-F_1(re^{it})w_p^+(re^{it})|^p dt<\varepsilon,\ 0<r<1. \tag{24}$$

Hence by Fatou theorem we have

$$\int_0^{2\pi} |B(e^{it})\psi_n(e^{it})w_p^+(e^{it})-F_1(e^{it})w_p^+(e^{it})|^p dt<\varepsilon,$$

i. e.

$$\int_0^{2\pi} |B(e^{it})\psi_n(e^{it})-F_1(e^{it})|^p \sigma'(t)dt<\varepsilon. \tag{25}$$

By virtue of (6), $\sigma'(t)\neq 0$ almost everywhere. By comparing (14) and (25) we have

$$F_1(e^{it})=f(e^{it})$$

almost everywhere.

Besides, from (24) it follows that

$$\int_0^{2\pi} |F_1(re^{it})w_p(re^{it})|^p dt\leqslant C.$$

Hence $F_1(z)w_p^+(z)\in H^p(|z|<1)$, $p>0$.

Now we are going to prove 3° and the second part of 1° in the definition of $L^p(\sigma(t);\{a_k\})$. The proof is similar to previous one.

Using (18) instead of (19) for each $\rho, 1<\rho<+\infty$, we have

$$\ln^+|\psi_n(\rho e^{i\varphi})|\leqslant\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-1}{1-2\rho\cos(t-\varphi)+\rho^2}\ln^+|\psi_n(e^{it})|dt\leqslant\frac{\rho+1}{\rho-1}C. \tag{26}$$

Hence the system of functions $\{\psi_n(z)\}, \psi_n(\infty)=0, n=1, 2, \cdots$, is a normal family in $|z|>1$. Consequently there exists a subsequence of $\{\psi_n(z)\}$ (without loss of generality we can assume it is just $\{\psi_n(z)\}$), which converges uniformly in the interior of

$|z|>1$ to some analytic function $F_3(z)$, $F_3(\infty)=0$:

$$\lim_{n\to+\infty}\psi_n(z)=F_3(z),\quad |z|>1. \tag{27}$$

Using (18) for each ρ, $1<\rho<+\infty$, we have

$$\int_0^{2\pi}\ln|F_3(\rho e^{it})|dt=\lim_{n\to+\infty}\int_0^{2\pi}\ln^+|\psi_n(\rho e^{it})|dt\leqslant\overline{\lim_{n\to+\infty}}\int_0^{2\pi}\ln^+|\psi_n(e^{it})|dt\leqslant C.$$

Hence $F_3(z)\in A(|z|>1)$. Consequently it has angular boundary values on $|z|=1$ almost everywhere.

From (12), (16) and (17) it follows that

$$\int_0^{2\pi}\left|\frac{\psi_n(e^{it})}{w_p^-(e^{it})}-\frac{\psi_{n+m}(e^{it})}{w_p^-(e^{it})}\right|^p dt<\varepsilon \tag{28}$$

and

$$\int_0^{2\pi}\left|\frac{\psi_n(e^{it})}{w_p^-(e^{it})}\right|^p dt\leqslant C. \tag{29}$$

Obviously, $\psi_n(z)\in H^p(|z|>1)$, $\psi_n(\infty)=0$. Consequently $\psi_n(z)\in D(|z|>1)$. Since $w_p^-(z)\in D(|z|>1)$, $w_p^-(z)\neq 0$ for $|z|>1$, hence

$$\frac{\psi_n(z)}{w_p^-(z)}\in D(|z|>1),\quad \frac{\psi_n(\infty)}{w_p^-(\infty)}=0,\ n=1,\ 2,\ \cdots. \tag{30}$$

From the boundary properties of analytic functions and (28) for each ρ, $1<\rho<+\infty$, we obtain

$$\int_0^{2\pi}\left|\frac{\psi_n(\rho e^{it})}{w_p^-(\rho e^{it})}-\frac{\psi_{n+m}(\rho e^{it})}{w_p^-(\rho e^{it})}\right|^p dt\leqslant\int_0^{2\pi}\left|\frac{\psi_n(e^{it})}{w_p^-(e^{it})}-\frac{\psi_{n+m}(e^{it})}{w_p^-(e^{it})}\right|^p dt<\varepsilon.$$

Let $m\to+\infty$. Using (27) from above inequality we have

$$\int_0^{2\pi}\left|\frac{\psi_n(\rho e^{it})}{w_p^-(\rho e^{it})}-\frac{F_3(\rho e^{it})}{w_p^-(\rho e^{it})}\right|^p dt<\varepsilon.$$

Hence by Fatau theorem we have

$$\int_0^{2\pi}\left|\frac{\psi_n(e^{it})}{w_p^-(e^{it})}-\frac{F_3(e^{it})}{w_p^-(e^{it})}\right|^p dt<\varepsilon. \tag{33}$$

Hence from (30) and (33)

$$\frac{F_3(z)}{w_p^-(z)}\in D(|z|>1)\quad \frac{F_3(\infty)}{w_p^-(\infty)}=0, \tag{34}$$

and

$$\int_0^{2\pi}|\psi_n(e^{it})-F_3(e^{it})|^p\sigma'(t)dt<\varepsilon,$$

i. e.

$$\int_0^{2\pi}|B(e^{it})\psi_n(e^{it})-B(e^{it})F_3(e^{it})|^p\sigma'(t)dt<\varepsilon. \tag{35}$$

By virtue of (6), $\sigma'(t)\neq 0$ almost everywhere. By comparing (14) and (35) we have

$$f(e^{it})=B(e^{it})F_3(e^{it}) \tag{36}$$

almost everywhere.

Let

$$F_2(z)=B(z)F_3(z)=B(z)w_p^-(z)\frac{F_3(z)}{w_p^-(z)}=B(z)w_p^-(z)\psi(z), \tag{37}$$

where $\psi(z)=\frac{F_3(z)}{w_p^-(z)}$.

From (29) for each ρ, $1<\rho<+\infty$, we have

$$\int_0^{2\pi}\left|\frac{\psi_n(\rho e^{it})}{w_p^-(\rho e^{it})}\right|^p dt\leqslant\int_0^{2\pi}\left|\frac{\psi_n(e^{it})}{w_p^-(e^{it})}\right|^p dt\leqslant C.$$

Consequently from (27) we have

$$\int_0^{2\pi}|\psi(\rho e^{it})|^p dt\leqslant\int_0^{2\pi}\left|\frac{F_3(\rho e^{it})}{w_p^-(\rho e^{it})}\right|^p dt\leqslant C. \tag{38}$$

Hence from the boundary properties of analytic functions and from (30) and (38) we obtain

$$\psi(z)\in H^p(|z|>1),\quad \psi(\infty)=0.$$

Besides, from (36) we see that

$$F_2(z)=f(z),\quad |z|=1,$$

holds almost everywhere.

Thus the proof of Theorem 1 is completed perfectly.

We denote by $H^p(\sigma(t);\ |z|<1)$ the collection of functions $f(z)$ which are analytic in $|z|<1$ and have the measurable angular boundary values on $|z|=1$ and satisfy the condition

$$\int_0^{2\pi}|f(e^{it})|^p d\sigma(t)<+\infty.$$

The norm of the space $H^p(\sigma(t);\ |z|<1)$ is defined as the norm of the space $L^p(\sigma(t);\ |z|=1)$.

Corollary 1. *The system of functions $\{\Omega_n(z)\}$ is not complete in the spaces $H^p(\sigma(t);|z|<1)$, $p>0$.*

Proof On the contrary if the system $\{\Omega_n(z)\}$ was complete in the spaces $H^p(\sigma(t);\ |z|<1)$, $p>0$, then according to the Theorem 1 and the definitions of norm in $H^p(\sigma(t);\ |z|<1)$, $p>0$, there would be a function $F_2(z)$ analytic in $|z|>1$ except at $\left\{\frac{1}{a_k}\right\}$, $k=1, 2, \cdots$, $F_2(\infty)=0$, such that

1° $f(z)=F_2(z)$ on $|z|=1$ almost everywhere,

2° $F_2(z)=B(z)\psi(z)w_p^-(z)$,

where

$$\psi(z)\in H^p(|z|>1),\quad \psi(\infty)=0,$$

and $B(z)$ is the Blaschke product defined by (2).

Generally it is impossible, hence Corollary 1 is valid.

From Corollary 1 we deduce immediatly the following Corollary 2.

Corollary 2. *The system of functions $\{\Omega_n(z)\}$ is not complete in $H^p(|z|<1)$.*

In fact

$$H^p(|z|<1)=H^p(t;|z|<1),$$

i. e. it is the case of $\sigma(t)=t$.

We have obtained this result already [3].

Now we want to obtain the inverse theorem under the condition $\sigma(t)=t,\ 0\leqslant t\leqslant 2\pi$.

Theorem 2. *The system of functions* $\{\Omega_n(z)\}$ *is complete in the spaces* $L^p(\sigma(t); \{a_k\}),\ p\geqslant 1$.

Proof According to the Hahn-Banach theorem it is sufficient to prove that for any linear functional $I(f)$ in $L^p(t; \{a_k\})$ from

$$I(\Omega_n(z))=0,\ n=1,\ 2,\ \cdots, \tag{39}$$

it follows that $I(f)\equiv 0$.

Since the space $L^p(t; \{a_k\})$ is the subspace of $L^p(t; |z|=1)$, every linear functional $I(f)$ has the integral representation

$$I(f)=\frac{1}{2\pi}\int_0^{2\pi}\overline{g(e^{it})}f(e^{it})dt,$$

where

$$g(e^{it})\in L^q(t;\ |z|=1),\ 1/q+1/p=1,\ p>1.$$

(Here and after we can assume $p>1$, since the case of $p=1$ can be studied by the similar method.)

Based on the definition of $L^p(t; \{a_k\})$, we know $f(z)\in H^p(|z|<1)$. For every fixed ζ, $|\zeta|>1$, we take the function $(z-\zeta)^{-1}\in L^p(t; \{a_k\})$, hence

$$\Phi(\zeta)\triangleq -I\left(\frac{1}{z-\zeta}\right)=-\frac{1}{2\pi i}\int_{|z|=1}\frac{\overline{g(z)}}{z-\zeta}\,\frac{dz}{z}.$$

By virtue of $\frac{\overline{g(z)}}{z}\in L^q(|z|=1),\ q>1$, we have by the Riesz theorem

$$\Phi(\zeta)\in H^q(|\zeta|>1),\ \Phi(\infty)=0. \tag{40}$$

Thus we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z|=1}\Phi(z)f(z)dz&=-\frac{1}{2\pi i}\int_{|z|=1}\left[\frac{1}{2\pi i}\int_{|\tau|=1}\frac{\overline{g(\tau)}}{\tau-z}\,\frac{d\tau}{\tau}\right]f(z)dz\\&=-\frac{1}{2\pi i}\int_{|\tau|=1}\frac{\overline{g(\tau)}}{\tau}\left[\frac{1}{2\pi i}\int_{|z|=1}\frac{f(z)dz}{\tau-z}\right]d\tau\\&=\frac{1}{2\pi i}\int_{|\tau|=1}\frac{\overline{g(\tau)}}{\tau}f(\tau)d\tau=\frac{1}{2\pi}\int_0^{2\pi}\overline{g(e^{it})}f(e^{it})dt,\end{aligned}$$

and

$$I(f)=\frac{1}{2\pi i}\int_{|z|=1}\Phi(z)f(z)dz,\ f(z)\in L^p(t;\{a_k\}). \tag{41}$$

From (39) we obtain

$$I(\Omega_n)=\frac{1}{2\pi i}\int_{|z|=1}\Phi(z)\Omega_n(z)dz=0,\ n=1,\ 2,\ \cdots. \tag{42}$$

Suppose

$$\frac{1}{2\pi i}\int_{|\tau|=1}\frac{\Phi(\tau)B(\tau)}{\tau-z}\,d\tau$$

determines two functions $\psi_+(z)$ and $\psi_-(z)$ analytic in $|z|<1$ and $|z|>1$ respectively, $\psi_-(\infty)=0$.

Using (42) and Theorem 1 in [3] we arrive at

$$\frac{1}{2\pi i}\int_{|\tau|=1}\frac{\Phi(\tau)}{(\tau-\bar{a}_k^{-1})^{s_k}}d\tau=0,\quad k=1,\ 2,\ \cdots.$$

Using (40) we have

$$\Phi^{(s_k-1)}(\bar{a}_k^{-1})=0,\ k=1,\ 2,\ \cdots.$$

Hence

$$\Phi(z)B(z)\in H^q(|z|>1),\ \Phi(\infty)B(\infty)=0. \tag{43}$$

Based on the definition of $L^p(t;\{a_k\})$, we have

$$I(f)=\frac{1}{2\pi i}\int_{|z|=1}\Phi(z)B(z)\psi(z)dz,$$

where $\psi(z)\in H^p(|z|>1)$, $\psi(\infty)=0$ and $B(z)$ is defined by (2).

By virtue of (43)

$$\Phi(z)B(z)\psi(z)\in H^1(|z|>1)$$

and $\Phi(z)B(z)\psi(z)$ has a zero at $z=\infty$ at least with its multiplicity 2. Hence using the Cauchy formula we obtain

$$I(f)\equiv 0 \text{ for } f\in L^p(t;\{a_k\}).$$

We completed the proof of Theorem 2.

From Theorems 1 and 2 we obtain immediatly Theorem 3.

Theorem 3. *The necessary and sufficient condition for the function $f(z)\in L^p(t, |z|=1)$, $p\geqslant 1$, to be approximated by the linear combination of system $\{\Omega_n(z)\}$ is $f(e^{it})\in L^p(t;\ \{a_k\})$*, i. e.

1° $f(z)\in H^p(|z|<1)$, (44)

2° $f(z)=B(z)\psi(z)$, $|z|>1|$, (45)

where $B(z)$ is defined by (2) *and* $\psi(z)\in H^p(|z|>1)$, $\psi(\infty)=0$.

Hence the closure of $\{\Omega_n(z)\}$ in $L^p(t;\ |z|=1)$, $p\geqslant 1$, is $L^p(t;\{a_k\})$.

We say that the sequence $\{a_k\}$ belongs to class $\Delta(P,\ \delta)$, if

$$\sup s_k=\sup p_k=P<+\infty \tag{46}$$

and

$$\inf_k\prod_{a_j\neq a_k}\left|\frac{a_j-a_k}{1-\bar{a}_k a_j}\right|\geqslant\delta>0. \tag{47}$$

Obviously if $\{a_k\}\in\Delta(P,\ \delta)$, then (1) is valid.

Theorem 4. *Let $\{a_k\}\in\Delta(P,\ \delta)$, $p>1$. Then the system of functions $\{\Omega_n(z)\}$ is the basis in the spcces $L^p(t;\{a_k\})$.*

Proof Let $f(e^{it})\in L^p(t;\ \{a_k\})$, it means that $f(z)$ satisfies the conditions (44) and (45). From the Lemma 2 in [4] we know that if $\{a_k\}\in\Delta(P,\ \delta)$, $f(z)\in H^p(|z|<1)$, then

$$f(z)=\sum_{k=1}^{+\infty}d_k(f)\Omega_k(z)+\frac{B(z)}{1\pi i}\int_{|\tau|=1}\frac{f(\tau)}{B(\tau)}\frac{1}{\tau-z}d\tau, \tag{48}$$

where

$$d_k(f)=\frac{1}{2\pi}\int_{|\tau|=1}f(\tau)\left(\overline{\frac{(s_k-1)!\tau^{\varkappa_k-1}}{(1-\bar{a}_k\tau)^{s_k}}}\right)|d\tau|=f^{(s_k-1)}(a_k),\quad k=1,\ 2,\ \cdots,$$

and the above series conveges in the spaces $L^p(t;\ |z|=1)$.

By comparing (48) and (45) we have

$$f(z)=\sum_{k=1}^{+\infty} d_k(f)\Omega_k(z),$$

this completes the proof of Theorem 4.

Thus the system $\{\Omega_n(z)\}$ is the basis in its clouse $L^p(t;\ a_k)$, $p>1$.

We will prove that for the case of $0<p\leqslant 1$ the system $\{\Omega_n(z)\}$ is not a basis in its closure which we denote by $l^p(t;\{a_k\})$,

Theorem 5.

1° *For any sequence* $\{a_k\}$ *the system* $\{\Omega(z)\}$ *is not a basis in its closure* $l^p(t;\{a_k\})$, $0<p<1$.

2° *Suppose* $\{a_k\}$ *is a monotone increasing sequence,* $0<a_k<1$, *and there exists infinite number of* n_j *such that*

$$(1-a_{n_j})\leqslant C(1-a_{n_j+1}),\ c>1. \tag{48}$$

Then the system $\{\Omega_n(z)\}$ *is not a basis in its closure* $l'(t;\{a_k\})$.

Proof Since $\Omega_n(z)\in L^p(t;\ \{a_k\})$, we have

$$l^p(t;\ \{a_k\})\subset L^p(t;\ \{a_k\}),\ 0<p\leqslant 1.$$

1° We construct a function, strictly monotone increasing and continuous on $[0,\ \pi]$, satisfying the following conditions:

$$w(0)=0,\ w'(t)=0 \text{ on } [0,\ \pi] \text{ almost everywhere.}$$

Let

$$\mu(t)=\begin{cases} w(t), & 0\leqslant t\leqslant \pi,\\ w(2\pi-t), & \pi\leqslant t\leqslant 2\pi,\end{cases}$$

and

$$G(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t),\quad |z|<1.$$

Clearly

$$G(0)=\frac{1}{2\pi}(\mu 2\pi)-\mu(0))=0$$

and

$$\mathrm{Re}G(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(t-\varphi)+r^2}\,d\mu(z),\ z=re^{i\varphi},\ 0\leqslant r<1.$$

Hence $\mathrm{Re}\, G(z)>0$, $|z|<1$,

$$\int_0^{2\pi}|\mathrm{Re}\ G(re^{i\theta})|\,d\theta<+\infty,\ 0\leqslant r<1, \tag{49}$$

and

$$G(z)\in H^p(|z|<1), \tag{50}$$

for each p, $0<p<1$,

$$\mathrm{Re}\ G(e^{i\theta})=0 \quad \text{almost everywhere.} \tag{51}$$

Let

$$H(z)=\begin{cases} H_+(z)=-G(z), & \text{for } |z|<1,\\ H_-(z)=\overline{G(\bar z^{-1})}, & \text{for } |z|>1.\end{cases}$$

Then from (51) it follows that

$$H_+(e^{i\theta})=H_-(e^{i\theta})$$

almost everywhere and

$$H_+(z)\in H^p(|z|<1)\quad H_-(z)\in H^p(|z|>1),\ H_-(\infty)=0,\ 0<p<1. \tag{52}$$

Let

$$L(z)=B(z)H_+(z),\ |z|<1,$$
$$S(z)=B(z)H_-(z),\ |z|>1.$$

It is obvious that

$$L(z)\in H^p(|z|<1)\quad 0<p<1.$$

Besides we have

$$\lim_{r\to 1-0, z=re^{i\theta}} L(z)=B(e^{i\theta})H_+(e^{i\theta})$$

and

$$\lim_{\rho\to 1+0\ z=\rho e^{i\theta}} S(z)=B(e^{i\theta})H_-(e^{i\theta})$$

almost everywhere. Hence

$$L(z)=S(z)$$

is valid on $|z|=1$ almost everywhere. It means

$$L(z)\in L^p(t;\ \{a_k\}),\ 0<p<1$$

and $L(z)\neq 0$

$$L^{(s_k-1)}(a_k)=0,\ k=1,\ 2,\ \cdots.$$

We will prove that the function $L(z)$ cannot be expanded in the series of system $\{\Omega_n(z)\}$.

On the contrary if

$$\lim_{n\to+\infty}\int_0^{2\pi}\left|L(e^{it})-\sum_{j=1}^{n}a_j\Omega_j(e^{it})\right|^p dt=0,$$

then using the same method as in the proof of Theorem 1 we arrive at

$$\lim_{n\to+\infty}\sum_{j=1}^{n}a_j\Omega_j(z)=L(z),$$

which converges in the interior of $|z|<1$ uniformly. Consequently for any a_k, $k=1,\ 2,\ \cdots$, using the Weierstras theorem we have

$$\lim_{n\to+\infty}\sum_{j=1}^{n}a_j\Omega_j^{(s_k-1)}(a_k)=L^{(s_k-1)}(a_k)=0.$$

By using (4) we obtain

$$a_k=0,\ k=1,\ 2,\ \cdots,$$

it means $L(z)\equiv 0$. So we are in the contradiction. Thus we have proved 1°.

2° We will use the main idea of [5] to prove 2°.

Let $\{w_k\}$ be the sequence in l^1, i. e.

$$\sum_{k=1}^{+\infty}|w_k|<+\infty. \tag{53}$$

Then for every k we can choose a subsequence $\{a_{n_k}\}$ of $\{a_k\}$ such that

$$\text{A.}\quad |w_k|\int_{|z|=1}\left|\frac{B(z)}{z-a_{n_k}}\right||dz|=|w_k|\int_{|z|=1}(1-\bar{a}_{n_k}z)^{-1}|dz|$$
$$\geqslant k+\int_{|z|-1}\left|\sum_{j=0}^{k-1}\frac{w_jB(z)}{z-a_{n_j}}-\frac{w_jB(z)}{z-a_{n_j+1}}\right||dz| \tag{54}$$

and

B. $$\int_{|z|=1}\left|\frac{B(z)}{z-a_{n_j}}-\frac{B(z)}{z-a_{n_j+1}}\right||dz|\leqslant C<+\infty. \tag{55}$$

In fact since

$$\int_{|z|=1}\left|\frac{1}{1-\bar{a}_l z}\right||dz|\to+\infty \quad (l\to+\infty),$$

for any fixed k if all $a_{n_1}, a_{n_1+1}, a_{n_2}, a_{n_2+1}, \cdots, a_{n_{k-1}}, a_{n_{k-1}+1}$ have been chosen we can choose a_{n_k} such that (54) is valid. Besides

$$\begin{aligned}
&\int_{|z|=1}\left|\frac{B(z)}{z-a_{n_j}}-\frac{B(z)}{z-a_{n_j+1}}\right||dz| \\
&=\int_{|z|=1}\left|\frac{z-a_{n_j+1}}{(1-\bar{a}_{n_j}z)(1-\bar{a}_{n_j+1}z)}-\frac{z-a_{n_j}}{(1-\bar{a}_{n_j+1}z)(1-\bar{a}_{n_j}z)}\right||dz| \\
&=\int_{|z|=1}\left|\frac{a_{n_j}-a_{n_j+1}}{(1-\bar{a}_{n_j}z)(1-\bar{a}_{n_j+1}z)}\right||dz|=\int_{|z|=1}\left|\frac{a_{n_j}}{1-a_{n_j}z}-\frac{a_{n_j+1}}{1-a_{n_j+1}z}\right||dz| \\
&\leqslant C\int_{-1}^{1}\left|\frac{a_{n_j}}{1-a_{n_j}x}-\frac{a_{n_j+1}}{1-a_{n_j+1}x}\right|dx,
\end{aligned} \tag{56}$$

where we have used the Lemma 3 in [5]. Obviously

$$\begin{aligned}
&\int_{-1}^{1}\left|\frac{a_{n_j}}{1-a_{n_j}x}-\frac{a_{n_j+1}}{1-a_{n_j+1}x}\right|dx=\int_{-1}^{1}\left(\frac{a_{n_j+1}}{1-a_{n_j+1}x}-\frac{a_{n_j}}{1-a_{n_j}x}\right)dx \\
&=\ln\frac{1+a_{n_j+1}}{1-a_{n_j+1}}-\ln\frac{1+a_{n_j}}{1-a_{n_j}}=\ln\frac{1-a_{n_j}}{1-a_{n_j+1}}-\ln\frac{1+a_{n_j+1}}{1+a_{n_j}}\leqslant C,
\end{aligned} \tag{57}$$

where we have used the condition (48).

Combining (56) and (57) we obtain (55).

Now we construct a series

$$G(z)=\sum_{k=1}^{+\infty}\left(\frac{w_k B(z)}{z-a_{n_k}}-\frac{w_k B(z)}{z-a_{n_k+1}}\right). \tag{59}$$

By virtue of (53) and (55), using the Cauchy criteria we deduce that the series (59) converges in the spaces $L'(t;\ |z|=1)$. Hence applying the same method as in the proof of Theorem 1 we can deduce

$$G(z)\in l'(t;\{a_k\})\subset L'(t;\{a_k\}).$$

On the contrary if the system $\{\Omega_n(z)\}$ were the basis in $l'(t;\ \{a_k\})$, then by applying (4), i. e.

$$\frac{1}{2\pi}\int_{|z|=1}\overline{\left(\frac{1}{1-a_k z}\right)}\Omega_n(z)|dz|=\delta_{n,k}=\begin{cases}0, & \text{for } n\neq k,\\ 1, & \text{for } n=k,\end{cases}$$

we would obtain

$$G(z)=\sum_{l=1}^{+\infty}\frac{u_l B(z)}{z-a_l},$$

which converges in the spaces $L'(t;\ |z|=)=L(|z|=1)$, where

$$u_l=\begin{cases}w_j, & \text{for } l=n_j,\\ -w_j, & \text{for } l=n_j+1,\\ 0, & \text{for other cases.}\end{cases}$$

Consider the partial sums

$$S'_{n_k}(z)=\sum_{l=1}^{n_k}\frac{u_l B(z)}{z-a_l}=\frac{w_k B(z)}{z-a_{n_k}}+\sum_{j=1}^{k-1}\left(\frac{w_j B(z)}{z-a_{n_j}}-\frac{w_j B(z)}{z-a_{n_j+1}}\right).$$

By using (54) we have

$$\int_{|z|=1}|S_{n_p}(z)||dz|\geqslant|w_k|\int_{|z|=1}\left|\frac{B(z)}{z-a_{n_k}}\right||dz|$$
$$-\int_{|z|=1}\left|\sum_{j=1}^{k-1}\frac{w_j B(z)}{z-a_{n_j}}-\frac{w_j B(z)}{z-a_{n_j+1}}\right||dz|\geqslant k\to+\infty.$$

It means that the $S_{n_k}(z)$ does not converge in the space $L(|z|=1)$. Thus we are in the contradiction.

The proos of Theorem 5 is completed.

References

[1] Privalov, I. I., Boundary properties of analytic, functions Moscow-Leningrad, 1950 (Russian).

[2] Dzarbasjan, M. M., Biorthogonal systems and the solution of interpolation problem, bases on the nodes with bounded multiplicity in class H_2, *Izv. Akad. Nauk Armjan SSR Ser. math.*, **9**: 5(1974), 339—373 (Russian).

[3] Shen, X. C., On the problem of incompleteness of the biorthogonal systems of functions, Approximation Theory and its Applications (to appear).

[4] Shen. X. C., On the basis of rational functions in certain class of domains, *Journal of Approxiamtion Theory and its Applications*, **1**:1 (1984), 123—140.

[5] Hairapetian H. M., On the representation by the system of rational functions for some subclasses of Hardy spaces, *Izv. Akad. Nauk Armjan SSR ser. Math.*, **15**: 1(1980), 3—14 (Russian).