Chin. Ann. of Math.
9B (3) 1988

GLOBALLY DEFINED CLASSICAL SOLUTIONS TO FREE BOUNDARY PROBLEMS WITH CHARACTERISTIC BOUNDARY FOR QUASILINEAR HYPERBOLIC SYSTEMS

Li Tatsien (李大潜)* Zhao Yanchun (赵彦淳)**

Abstract

In this paper, the authors prove the global existence and uniqueness of classical solutions to some free boundary problems with characteristic boundary for the reducible quasilinear hyperbolic system.

§ 1. Introduction

Up to now, most of studies concerning the global existence of classical solutions to quasilinear hyperbolic systems are concentrated on the Cauchy problem for the reducible hyperbolic system

$$\begin{cases} \frac{\partial r}{\partial t}+\lambda(r, s)\frac{\partial r}{\partial x}=0, \\ \frac{\partial s}{\partial t}+\mu(r, s)\frac{\partial s}{\partial x}=0, \end{cases} \tag{1.1}$$

for boundary value problems, especially for free boundary problems, however, there are only few results.

Suppose that system (1.1) is strictly hyperbolic on the clomain under consideration:

$$\lambda(r, s)<\mu(r, s), \tag{1.2}$$

and satisfies the following conditions:

$$\frac{\partial \lambda}{\partial r}(r, s)\geqslant 0, \quad \frac{\partial \mu}{\partial s}(r, s)\geqslant 0. \tag{1.3}$$

Particularly, system (1.1) is genuinely nonlinear in the sense of P. D. Lax, if strict inequalities hold in (1.3). In this paper, under appropriate assumptions of monotonicity, we prove the global existence and uniqueness of classical solutions to some typical boundary value problems and free boundary problems with charac-

Manuscript received February 27, 1986, Revised March 17, 1987.
* Department of Mathematics and Institute of Mathematics, Fudan University, Shanghai, China.
** Institute of Mathematics, Fudan University, Shanghai, China.

teristic boundary for system (1.1) on an angular domain. These results can be used to discuss some kind of discontinuous initial value problems for the system of one-dimensional isentropic flow so that we can prove, unper certain hypotheses, the existence or the nonexistence of globally defined discontinuous solutions containing only one shock in a class of piecewise continuous and piecewise smooth functions, and get the corresponding results to the interaction problem of a typical shock with a rarefaction wave. Since the space is limited, we shall give all the details of these applications in a forthcoming paper (see [1]).

§ 2. Globally Defined Classical Solutions to a Class of Typical Boundary Value Problems with Characteristic Boundary

On an angular domain

$$R=\{(t,\ x)\,|\,t\geqslant 0,\ x_1(t)\leqslant x\leqslant x_2(t)\}, \tag{2.1}$$

we consider the following typical boundary value problem with characteristic boundary for system (1.1):

$$\text{on } x=x_1(t),\ s=s_0(t), \tag{2.2}$$

$$\text{on } x=x_2(t),\ r=g(t,\ s). \tag{2.3}$$

Here, $x=x_1(t)$ and $x=x_2(t)$ are all given curves passing through the origin with

$$x_1(0)=x_2(0)=0,\ x_1(t)<x_2(t),\ \forall t>0, \tag{2.4}$$

moreover, $x=x_1(t)$ is a backward characteristic curve on which we have

$$r=r_0\triangleq g(0,\ S_0), \tag{2.5}$$

$$x_1'(t)=\lambda(r_0,\ s_0(t)), \tag{2.6}$$

where

$$S_0=s_0(0). \tag{2.7}$$

We give the following hypotheses:

(H1) On the domain under consideration, λ, μ, s_0, $g\in C^1$, $x_1(t)$, $x_2(t)\in C^2$.

(H2) On $x=x_2(t)$, the following a priori estimates hold:

$$\lambda(r,\ s)<x_2'(t)<\mu(r,\ s) \tag{2.8}$$

and

$$x_2'(t)-\lambda(r,\ s)\geqslant a(T_0,\ A,\ B)>0,\ \forall 0\leqslant t\leqslant T_0,\ \forall |r|\leqslant A,\ \forall |s|\leqslant B, \tag{2.9}$$

where $a(T_0,\ A,\ B)$ denotes a constant depending only on T_0, A and B.

(H3) We have

$$s_0'(t)\leqslant 0,\ \forall t\geqslant 0 \tag{2.10}$$

and the following a priori estimate:

$$\text{on } x=x_2(t),\ \frac{\partial g}{\partial t}(t,\ s)\geqslant 0,\ \ \frac{\partial g}{\partial s}(t,\ s)\leqslant 0. \tag{2.11}$$

Theorem 1. *Suppose that* (1.2)—(1.3) *hold, under hypotheses* (H1)—(H3), *the typical boundary value problem with characteristic boundary* (1.1), (2.2)—(2.7) *admits a unique global* C^1 *solution* $(r(t, x), s(t, x))$ *on the angular domain* R. *Moreover, we have*

$$\frac{\partial r(t, x)}{\partial x}\geqslant 0, \quad \frac{\partial s(t, x)}{\partial x}\geqslant 0, \quad \forall (t, x)\in R. \tag{2.12}$$

Proof According to the corresponding theorem on the local existence and uniqueness (see Chapter 3 in [2]), by (H1) and

$$\lambda(r_0, S_0)<x_2'(0)<\mu(r_0, S_0) \tag{2.13}$$

(which follows directly from (2.8) by setting $t=0$), there exists a positive number $\delta_0>0$ such that this problem admits a unique local C^1 solution on the angular domain

$$R(\delta_0)=\{(t, x)\,|\,0\leqslant t\leqslant\delta_0,\ x_1(t)\leqslant x\leqslant x_2(t)\}. \tag{2.14}$$

In order to get a global C^1 solution on R, it is only necessary to establish the following uniform a priori estimate:

For any given $T_0>0$, if this problem possesses a C^1 solution $(r(t, x), s(t, x))$ on an angular domain

$$R(T)=\{(t, x)\,|\,0\leqslant t\leqslant T,\ x_1(t)\leqslant x\leqslant x_2(t)\} \tag{2.15}$$

with $0<T\leqslant T_0$, then the C^1 norm of the solution has an upper bound depending only on T_0 (independent of T).

We first estimate the C^0 norm of $(r(t, x), s(t, x))$ on $R(T)$.

By (2.6) and (2.8), the backward (resp. forward) characteristic curve passing through the point $(t, x)\in R(T)$ must intersect the boundary curve $x=x_2(t)$ (resp. $x=x_1(t)$) at one and only one point, denoted by $(\beta(t, x), \eta(t, x))\triangleq(\beta(t, x), x_2(\beta(t, x)))$ (resp. $(\alpha(t, x), \xi(t, x))\triangleq(\alpha(t, x), x_1(\alpha(t, x)))$. Then, by means of boundary conditions (2.2)—(2.3), it is easy to see that

$$s(t, x)=s_0(\alpha(t\ x)), \tag{2.16}$$

$$r(t, x)=g(\beta(t, x), s(\beta(t, x), \eta(t, x))). \tag{2.17}$$

Noting that

$$\alpha(t, x)\leqslant t, \tag{2.18}$$

it follows from (2.16) that

$$|s(t, x)|\leqslant C(T_0), \forall (t, x)\in R(T) \quad (0<T\leqslant T_0), \tag{2.19}$$

here and hereafter $C(T_0)$ stands for a constant depending only on T_0. Noticing that

$$\beta(t, x)\leqslant t, \tag{2.20}$$

(2.17) and (2.19) give

$$|r(t, x)|\leqslant C(T_0), \forall (t, x)\in R(T) \quad (0<T\leqslant T_0). \tag{2.21}$$

Thus, the uniform boundedness of the solution itself is obtained.

We now turn to the estimate of first derivatives of the solution. By system (1.1), it is only necessary to estimate the C^0 norm of $\partial r/\partial x(t, x)$ and $\partial s/\partial x(t, x)$ on $R(T)$.

The C^0 norm of $\partial s/\partial x(t, x)$ can be first estimated. Setting

$$v=e^{k(r,s)}\frac{\partial s}{\partial x}, \tag{2.22}$$

where $k(r, s)$ is defined by

$$\frac{\partial k}{\partial r}=\frac{\frac{\partial \mu}{\partial r}}{\mu-\lambda}, \tag{2.23}$$

it is easily seen that v satisfies the following Riccati's equation

$$\frac{dv}{dt}\triangleq\frac{\partial v}{\partial t}+\mu(r, s)\frac{\partial v}{\partial x}=-\frac{\partial \mu}{\partial s}(r, s)e^{-k(r,s)}v^2 \tag{2.24}$$

along the forward characteristic curve. Moreover, by means of (2.6) and the second equation in (1.1), it follows from (2.2) that

$$\text{on } x=x_1(t),\ v=\frac{e^{k(r_0, s_0(t))}}{(\lambda-\mu)(r_0, s_0(t))}s_0'(t). \tag{2.25}$$

By integration, from (2.24)—(2.25) we obtain

$$v(t, s)=\frac{e^{k(r_0, s_0(\alpha))}s_0'(\alpha)}{\begin{pmatrix}(\lambda-\mu)(r_0, s_0(\alpha)) \\ +\int_\alpha^t \frac{\partial \mu}{\partial s}(r(\tau, x(\tau, \alpha)), s_0(\alpha))s_0'(\alpha)e^{k(r_0, s_0(\alpha))-k(r(\tau, x(\tau,\alpha)), s_0(\alpha))}d\tau\end{pmatrix}}, \tag{2.26}$$

where $\alpha=\alpha(t, x)$, and $x=x(\tau, \alpha)$ denotes the forward characteristic passing through the point $(\alpha, x_1(\alpha))=(\alpha, \xi)$. Hence, noticing (2.10) and (1.2)—(1.3), we can immediately get the uniform boundedness of v and then $\partial s/\partial x$:

$$\left|\frac{\partial s}{\partial x}(t, x)\right|\leqslant C(T_0),\ \forall(t, x)\in R(T)\quad (0<T\leqslant T_0). \tag{2.27}$$

Furthermore, we have

$$\frac{\partial s}{\partial x}(t, x)\geqslant 0,\ \forall(t, x)\in R(T). \tag{2.28}$$

In order to estimate the C^0 norm of $\partial r/\partial x$ on $R(T)$, we first derive the following property:

$$\frac{\partial r}{\partial x}(t, x)\geqslant 0,\ \forall(t, x)\in R(T). \tag{2.29}$$

To do this, differentiating boundary condition (2.3) with respect to t and using system (1.1), we get

$$\text{on } x=x_2(t),\quad \frac{\partial r}{\partial x}=\frac{1}{x_2'(t)-\lambda}\left[\frac{\partial g}{\partial t}+\frac{\partial g}{\partial s}(x_2'(t)-\mu)\frac{\partial s}{\partial x}\right]. \tag{2.30}$$

Then, it follows from (2.8), (2.11) and (2.28) that

$$\text{on } x=x_2(t),\quad \frac{\partial r}{\partial x}\geqslant 0. \tag{2.31}$$

Since r must be a constant along every backward characteristic, we have

$$r(t, x)=r(\beta(t, x), \eta(t, x)),\ \forall(t, x)\in R(T), \tag{2.32}$$

where $\eta(t, x)=x_2(\beta(t, x))$. Then, noting system (1.1), we obtain

$$\frac{\partial r}{\partial x}(t, x)=\frac{\partial r}{\partial t}(\beta, \eta)\frac{\partial \beta}{\partial x}(t, x)+\frac{\partial r}{\partial x}(\beta, \eta)\frac{\partial \eta}{\partial x}(t, x)$$
$$=(x_2'(\beta(t, x))-\lambda(r(\beta, \eta), s(\beta, \eta)))\frac{\partial \beta}{\partial x}(t, x)\frac{\partial r}{\partial x}(\beta, \eta), \tag{2.33}$$

where $\beta=\beta(t, x)$ and $\eta=\eta(t, x)$. Since

$$\frac{\partial \beta}{\partial x}(t, x)\geqslant 0 \tag{2.34}$$

by the definition of $\beta(t, x)$ (we have actually the strict inequality in (2.34)), (2.29) follows directly from (2.8) and (2.31).

We now take care of the estimation of the C^0 norm of $\partial r/\partial x$ on $R(T)$. Similarly to (2.22), let

$$u=e^{h(r, s)}\frac{\partial r}{\partial x}, \tag{2.35}$$

where $h(r, s)$ is defined by

$$\frac{\partial h}{\partial s}=\frac{\dfrac{\partial \lambda}{\partial s}}{\lambda-\mu}. \tag{2.36}$$

It is easy to see that u satisfies the following Riccati's equation

$$\frac{du}{dt}\triangleq\frac{\partial u}{\partial t}+\lambda(r, s)\frac{\partial u}{\partial x}=-\frac{\partial \lambda}{\partial r}(r, s)e^{-h(r, s)}u^2 \tag{2.37}$$

along the backward characteristic. Moreover, at the point $(\beta(t, x), \eta(t, x))=(\beta(t, x), x_2(\beta(t, x)))$ on $x=x_2(t)$,

$$u=d^{h(r(\beta, \eta), s(\beta, x))}\frac{\partial r}{\partial x}(\beta, \eta), \tag{2.38}$$

where $\beta=\beta(t, x)$ and $\eta=\eta(t, x)$. Thus, by integration we get

$$u(t, x)=\frac{e^{h(r(\beta, \eta), s(\beta, \eta))}\dfrac{\partial r}{\partial x}(\beta, \eta)}{1+\displaystyle\int_\beta^t \frac{\partial \lambda}{\partial r}(r(\beta, \eta), s(\tau, \hat{x}(\tau, \beta)))e^{h(r(\beta, \eta), s(\beta, \eta))-h(r(\beta, \eta), s(x, \hat{x}(r, \beta)))}\frac{\partial r}{\partial x}(\beta, \eta)d\tau}, \tag{2.39}$$

where $x=\hat{x}(\tau, \beta)$ denotes the backward characteristic passing through the point (β, η). Therefore, noticing (1.3) and (2.31), in order to obtain a uniform estimate for u on $R(T)$, it is sufficient to establish a uniform estimate for the value of $\partial r/\partial x$ on $x=x_2(t)$. Noting (2.9) and the uniform estimates previously established for r, s and $\partial s/\partial x$, it comes from (2.30) that

$$\text{on } x=x_2(t), \quad \left|\frac{\partial r}{\partial x}\right|\leqslant C(T_0), \tag{2.40}$$

and then

$$\left|\frac{\partial r}{\partial x}(t, x)\right|\leqslant C(T_0), \ \forall (t, x)\in R(T) \quad (0<T\leqslant T_0). \tag{2.41}$$

The proof of Theorem 1 is complete.

Remark 1. In the special case

$$s_0(t) \equiv S_0, \tag{2.42}$$

by (2.6), $x = x_1(t)$ must be a straight characteristic and it holds for the solution in question that

$$s(t,\ x) \equiv S_0, \tag{2.43}$$

namely, the solution must be a backward simple wave. In this case, it is easy to see from (2.30) that only the first hypothesis is needed in (2.11). Moreover, let $r = \bar{r}_0(t)$ be the value of r on $x = x_2(t)$. By means of (2.8), (2.29) and system (1.1), we have

$$\bar{r}_0'(t) = (x_2'(t) - \lambda(r,\ s))\frac{\partial r}{\partial x}(t,\ x) \geqslant 0. \tag{2.44}$$

Then, by (1.3), this backward simple wave must be a rarefaction wave. Therefore, the problem still admits a unique global C^1 solution on R without hypothesis (2.9).

Remark 2. Suppose furthermore that the characteristic $\lambda(r,\ s)$ of system (1.1) is genuinely nonlinear:

$$\frac{\partial \lambda}{\partial r}(r,\ s) > 0. \tag{2.45}$$

If problem (1.1), (2.2)—(2.7) possesses a bounded, globalt C^1 solution $(r(t,\ x),\ s(t,\ x))$, then

$$\bar{r}_0(t) \triangleq r(t,\ x_2(t)) = g(t,\ s(t,\ x_2(t))) \tag{2.46}$$

must be a nondecreasing function of t.

In fact, similarly to (2.26), we have

$$u(t,\ x) = \frac{e^{h(r_0(\beta),\ s(\beta,\ \eta))}\bar{r}_0'(\beta)}{\left(\begin{array}{c}(x_2'(\beta) - \lambda(r_0(\beta),\ s(\beta,\ \eta)))\\ +\displaystyle\int_\beta^t \frac{\partial \lambda}{\partial r}(r_0(\beta),\ s(\tau,\ \hat{x}(\tau,\ \beta)))\bar{r}_0'(\beta)e^{h(r_0(\beta),\ s(\beta,\ \eta)) - \eta(r_0(\beta),\ s(r,\ \hat{x}(r,\ \beta)))}d\tau\end{array}\right)}, \tag{2.47}$$

where $x = \hat{x}(\tau,\ \beta)$ denotes the backward characteristic curve passing through the point $(\beta,\ \eta)$, where $\beta = \beta(t,\ x)$ and $\eta = \eta(t,\ x) = x_2(\beta(t,\ x))$. Thus, if there exists a β such that $\bar{r}_0'(\beta) < 0$, then the backward characteristic $x = \hat{x}(\tau,\ \beta)$ passing through the point $(\beta,\ \eta)$ can be infinitely extended, since $x = x_1(t)$ is also a backward characteristic. By means of the boundedness of the solution and the genuinely nonlinear hypothesis (2.45), the denominator in (2.47) must change the sign in a finite time along this characteristic $x = \hat{x}(\tau,\ \beta)$, hence, it is impossible to have a global C^1 solution on R. It turns out that $\bar{r}_0(t)$ defined by (2.46) must be a nondecreasing function of t. This fact can be used to show the nonexistence of global C^1 solutions in certain cases (see [1]).

In the special case that g does not explicitly depend on s, it can be then seen that, to guarantee the global existence of C^1 solutions, hypothesis of monotonicity (2.11) is actually necessary.

§ 3. Globally Defined Classical Solutions to a Class of Typical Free Boundary Problems with Characteristic Boundary

We now generalize the result obtained in § 2 to the corresponding case of free boundary problems.

On an angular domain

$$R=\{(t, x) \mid t\geqslant 0,\ x_1(t)\leqslant x\leqslant x_2(t)\}, \tag{3.1}$$

we consider the following typical free boundary problem with characteristic boundary for system (1.1):

$x=x_2(t)$ is a free boundary on which we prescribe the boundary conditions

$$r=g(t, x, s), \tag{3.2}$$

$$x_2'(t)=G(t, x, s),\ x_2(0)=0; \tag{3.3}$$

on the other hand, $x=x_1(t)$ is a given backward characteristic passing through the origin, on which we prescribe the boundary condition

$$s=s_0(t). \tag{3.4}$$

Moreover, on $x=x_1(t)$ we have

$$r=r_0\triangleq g(0,\ 0,\ S_0) \tag{3.5}$$

and

$$x_1'(t)=\lambda(r_0,\ s_0(t)),\ x_1(0)=0, \tag{3.6}$$

where

$$S_0=s_0(0). \tag{3.7}$$

We give the following hypotheses:

(H1) On the domain under consideration, λ, μ, s_0, g and $G\in C^1$, $x_1(t)\in C^2$.

(H2) On $x=x_2(t)$, the following a priori estimates hold:

$$\lambda(r, s)<G(t, x, s)<\mu(r, s) \tag{3.8}$$

and

$$G(t, x, s)-\lambda(r, s)\geqslant a(T_0, A, B)>0,\ \forall 0\leqslant t\leqslant T,\ \forall |r|\leqslant A,\ \forall |s|\leqslant B, \tag{3.9}$$

where $a(T_0, A, B)$ stands for a constant only depending on T_0, A and B.

(H3) We have

$$s_0'(t)\leqslant 0,\ \forall t\geqslant 0 \tag{3.10}$$

and the following a priori estimate

$$\text{on } x=x_2(t),\ \frac{\partial g}{\partial t}(t, x, s)+\frac{\partial g}{\partial x}(t, x, s)G(t, x, s)\geqslant 0,\ \frac{\partial g}{\partial s}(t, x, s)\leqslant 0 \tag{3.11}$$

(H4) The following a priori estimate holds on $x=x_2(t)$:

$$|G(t, x, s)|\leqslant a_1(T_0, B)+a_2(T_0, B)|x|,\ \forall 0\leqslant t\leqslant T_0,\ \forall |s|\leqslant B, \tag{3.12}$$

where $a_1(T_0, B)$ and $(a_2(T_0, B)$ denote constants depending only on T_0 and B.

Theorem 2. *Suppose that* (1.2)—(1.3) *hold, under hypotheses* (H1)—(H4), *the typical free boundary problem with characteristic boundary* (1.1), (3.2)—(3.7) *admits a unique globally defined classical solution*: $(r(t, x), s(t, x)) \in C^1(R)$ *and* $x_2(t) \in C^2$, $\forall t \geqslant 0$, *on the angular domain* R. *Moreover, we have*

$$\frac{\partial r}{\partial x}(t, x) \geqslant 0, \quad \frac{\partial s}{\partial x}(t, x) \geqslant 0, \ \forall (t, x) \in R. \tag{3.13}$$

Proof According to the corresponding theorem on the local existence and uniqueness (see Chapter 3 in [2]), by (H1) and

$$\lambda(r_0, S_0) = x_1'(0) < G(0, 0, S_0) = x_2'(0) < \mu(r_0, S_0) \tag{3.14}$$

(which comes directly from (3.8) by setting $t=0$), there exists a positive number $\delta_0 > 0$ such that this problem possesses a unique locally defined classical solution: $(r(t, x), s(t, x)) \in C^1$ and $x_2(t) \in C^2$ on the angular domain

$$R(\delta_0) = \{(t, x) \mid 0 \leqslant t \leqslant \delta_0, \ x_1(t) \leqslant x \leqslant x_2(t)\}. \tag{3.15}$$

In order to obtain the global existence of classical solution on R, it is only necessary to establish the following uniform a priori estimate:

For any given $T_0 > 0$, if, on an angular domain

$$R(T) = \{(t, x) \mid 0 \leqslant t \leqslant T, \ x_1(t) \leqslant x \leqslant x_2(t)\} \tag{3.16}$$

with $0 < T \leqslant T_0$, this free boundary problem admits a classical solution: $(r(t, x), s(t, x)) \in C^1$ and $x_2(t) \in C^2$, then the C^1 norm of $(r(t, x), s(t, x))$ has an upper bound depending only on T_0 (independent of T).

This uniform a priori estimate can be obtained in a similar manner as in § 2. As a matter of fact, if this free boundary problem admits a classical solution: $(r(t, x), s(t, x)) \in C^1$ and $x_2(t) \in C^2$ on $R(T)$, let

$$\hat{g}(t, s) = g(t, x_2(t), s), \tag{3.17}$$

$(r(t, x), s(t, x))$ can be described as a classical solution to the corresponding typical boundary value problem with characteristic boundary on $R(T)$, where $x = x_2(t)$ is regarded as a given curve with the following boundary condition

$$\text{on } x = x_2(t), \ r = \hat{g}(t, s). \tag{3.18}$$

By hypothesis of monotonicity (3.11), we have

$$\frac{\partial \hat{g}(t, s)}{\partial t} \geqslant 0, \quad \frac{\partial \hat{g}(t, s)}{\partial s} \leqslant 0. \tag{3.19}$$

It follows then that hypotheses (H1)—(H3) in Theorem 1 are all satisfied for this typical boundary value problem with characteristic boundary. Completely repeating the proof of Theorem 1, it is easy to see that, after having got the uniform estimate (2.19) for s, it is only necessary to prove the following uniform estimate for the free boundary $x = x_2(t)$:

$$|x_2(t)| \leqslant C(T_0), \ \forall 0 \leqslant t \leqslant T \quad (0 < T \leqslant T_0). \tag{3.20}$$

By (3.3), we have

$$x_2(t) = \int_0^t G(\tau, x_2(\tau), s(\tau, x_2(\tau))) d\tau. \tag{3.21}$$

Noting (3.12) and (2.19), we get

$$|x_2(t)| \leqslant C_1(T_0) + C_2(T_0)\int_0^t |x_2(\tau)|\, d\tau, \quad \forall 0 \leqslant t \leqslant T, \tag{3.22}$$

then the Gronwall's inequality gives the desired estimate (3.20).

The proof of Theorem 2 is complete.

Remark 3. Similar to Remark 1, in the special case that (2.42) holds, Theorem 2 is still valid without hypothesis (3.9) and the second hypothesis in (3.11). In fact, (2.43) still holds on the existence domain of the solution so that the solution must be a backward simple wave. Thus, (3.3) reduces to

$$x_2'(t) = G(t, x_2, S_0), \quad x_2(0) = 0, \tag{3.23}$$

and then, by (H4), the free boundary curve $x = x_2(t)$ can be globally predetermined. Moreover, by (3.2)—(3.3) and the first hypothesis in (3.11), it holds on $x = x_2(t)$ that

$$\frac{dr}{dt} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} G \geqslant 0. \tag{3.24}$$

Hence, by (1.3), this backward simple wave must be a rarefaction wave so that it is not necessary to get a uniform estimate for the C^0 norm of $\partial r/\partial x$ in order to construct the global solution. Moreover, $x = x_1(t)$ must be a straight characteristic in this case.

We shall use the result just mentioned above to get the existence of globally defined discontinuous solutions only containing one shock for a class of discontinuous initial value problems for the system of isentropic flow (see [1]).

Remark 4. Similarly to Remark 2, suppose furthermore that the characteristic $\lambda(r, s)$ is genuinely nonlinear, namely, (2.45) holds. If the free boundary problem (1.1), (3.2)—(3.7) possesses a globally defined classical solution on the angular domain R, and $(r(t, x), s(t, x))$ is bounded, then

$$\bar{r}_0(t) \triangleq r(t, x_2(t)) = g(t, x_2(t), s(t, x_2(t))) \tag{3.25}$$

must be a nondecreasing function of t. Therefore, in the case that g does not explicitly depend on x and s, hypothesis of monotonicity (3.11) is also necessary for guaranteeing the global existence.

Particularly, suppose that the value of $\partial r/\partial x$ at the origin, which is uniquely determined by system (1.1) and the boundary conditions, is negative:

$$\frac{\partial r}{\partial x}(0, 0) < 0. \tag{3.26}$$

Suppose furthermore that $s_0(t)$ is a bounded function of t and $g(t, x, s)$ is bounded as s is bounded. Then, the classical solution of this free boundary problem must blow up in a finite time. In fact, in this case the solution $(r(t, x), s(t,x))$ is bounded, however, $\bar{r}_0(t)$ defined by (3.25) cannot be a nondecreasing function of t. It must be observed that the conclusion is still valid even without hypotheses (H3) and (3.9).

We shall use the foregoing result to prove the nonexistence of globally defined discontinuous solutions containing only one shock for a class of discontinuous initial value problems for the system of isentropic flow (see [1]).

References

[1] Li Tatsien (Li Da-qian) and Zhao Yanchun, Global discontinuous solutions to a class of discontinuous initial value problems for the system of isentropic flow and applications (to appear in Chin. Ann. of Math.).

[2] Li Tatsien (Li Da-qian) and Yu Wen-ci, Boundary value problemsa for quasilinear hyperbolic systems, Duke University Mathematics Series V, 1985.