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EXISTENCE AND NON-EXISTENCE OF GLOBAL SMOOTH SOLUTIONS FOR QUASILINEAR HYPERBOLIC SYSTEMS*

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Abstract

Consider initial value probiom $v_t-u_x=0,\ u_t+p(v)_x=0$, (E), $v(x, 0)=v_0(x),\ u(x, 0)=u_0(x)$, (I), where $A\geqslant 0$, $p(v)=K^2v^{-\gamma}$, $K>0$, $0<\gamma<3$. As $0<\gamma\leqslant 1$, the authors give a sufficient condition for that (E), (I) to have a unique global smooth solution. As $1\leqslant\gamma<3$, a necessary condition is given for that.

§ 1. Introduction

The first order quasilinear system

$$v_t-v_x=0,\ u_t+p(v)_x=-2Au,\quad (A\geqslant 0) \tag{E}$$

arises in a variety of ways in several areas of applied mathematics. There has been much investigation to the problem of establishing global existence and non-existence theorems of smooth solutions for initial-boundary value problems.

Nishida[1] has considered the initial value problem for the damped quasilinear hyperbolic system (E) $(p'(v)<0)$ with smooth initial data

$$v(x, 0)=v_0(x),\ u(x, 0)=v_0(x), \tag{I}$$

which is small with respect to $A>0$. He has shown that initial value problem for (E), (I) will possess a unique global smooth solution. In other words, under some conditions, there exists $\varepsilon>0$, such that if

$$\sup_x|r(x, 0)|+\sup_x|s(x, 0)|+\sup_x|r_x(x, 0)|+\sup_x|s_x(x, 0)|<\varepsilon,$$

where r, s are Riemann invariants, then the initial value problem (E), (I) has a unique smooth solution in the large time. Here $\varepsilon>0$ depends on A, and $\varepsilon(A)\to 0$, as $A\to 0$.

Slemrod[2,3], on the other hand, has shown that if $|r(x, 0)|$ $|s(x, 0)|$ are

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sufficiently small and $r_x(x, 0)$ or $s_x(x, 0)$ is negative and sufficiently large in the absolute value at any point $x \in R$, then the initial value problem for (E), (I) has a C^1-solution only for a finite time.

It is more interesting to investigate the problem with "big" initial value.

In the case $A=0$[6-11] the necessary and sufficient condition of the existence for global smooth solution is

$$r_0'(x) \geqslant 0, \quad s_0'(x) \geqslant 0.$$

The condition is available for "big" initial data.

In section 2 of this paper, we give some conditions for existence or nonexistence of global smooth solution for certain $p(v)$. The conditions are available for $A \geqslant 0$ and "big" initial data.

In detail, suppose that $p(v)=K^2 v^{-\gamma}$, $K>0$, $0<\gamma<3$, $v>0$. Introduce Riemann invariants as $r=u+\Phi(v)$, $s=u-\Phi(v)$, where $\Phi(v)=\int_1^v \sqrt{-p'(s)}ds$. The main result in section 2 can be stated as follows:

Under conditions (A), (C),

$$\sup_x |r_0(x)| + \sup_x |s_0(x)| < \min\{-2\Phi(0), 2\Phi(\infty)\}. \tag{A}$$

$$r_0(x),\ s_0(x) \in C^1(R),\ |r_0'(x)| + |s_0'(x)| \leqslant M, \tag{C}$$

(1) As $0<\gamma\leqslant 1$, if condition (B)

$$r_0'(x) \geqslant -\frac{4A}{3-\gamma} v_0(x),\ s_0'(x) \geqslant -\frac{4A}{3-\gamma} v_0(x) \tag{B}$$

holds for all $x \in R$, then the initial value problem (E), (I) has a unique global smooth solution.

(2) As $1 \leqslant \gamma < 3$, if condition (B) fails at any $x \in R$, then the initial value problem (E), (I) has a C^1-solution for only a finite time.

Combining (1), (2), we obtain a necessary and sufficient condition of the existence of global smooth solutions for $\gamma=1$. This is the result of Zheng in [5].

The condition (A) ensures that the C^1-solution is strictly away from $v=\infty$ (as $1<\gamma<3$) or $v=0$ (as $0<\gamma<1$). In view of the case $A=0$ (see [10, 11]) we believe that the right side of the inequality (A) can be replaced by ∞.

§2. The Existence and Nonexistence of Global Smooth Solutions

Consider the initial value problem (E), (I), where $p(v)=K^2 v^{-\gamma}$, $K>0$, $0<\gamma<3$, $v>0$. The characteristics are

$$\lambda = -\sqrt{-p'(v)} = -b^{-2} v^{-\frac{\gamma+1}{2}},\ \mu = \sqrt{-p'(v)} = b^{-2} v^{-\frac{\gamma+1}{2}},$$

where $b=K^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}$, and the Riemann invariants are taken as

$$r=u+\Phi(v), \quad s=u-\Phi(v),$$

$$\Phi(v)=\int_1^v \mu(s)ds.$$

The Rimann invariants give a one to one smooth mapping from $\Omega=\{(u, v)\,|\,u\in R, v\in R_+\}$ onto $\Omega_1=\{(r, s)\,|\,2\Phi(0)<r-s<2\Phi(\infty)\}$. The Riemann invariants diagonalize the principal parts of the system (E) as

$$r_t+\lambda r_x=-A(r+s), \quad s_t+\mu s_x=-A(r+s) \tag{E$_1$}$$

and transform the initial data (I) into

$$\begin{aligned} r(x, 0)&=r(u_0(x), v_0(x))=r_0(x),\\ s(x, 0)&=s(u_0(x), v_0(x))=s_0(x), \end{aligned} \tag{I$_1$}$$

We borrow a priori estimate of Nishida[1].

Lemma 2.1. *Under the conditions* (A) (C), *the initial value problem* $(E)_1$, $(I)_1$ *has the a priori estimate for C^1-solution:*

$$\sup_x|r(x, t)|+\sup_x|s(x, t)|\leqslant\sup_x|r_0(x)|+\sup_x|s_0(x)|\leqslant\min\{-2\Phi(0), 2\Phi(\infty)\},$$

for $t\geqslant 0$ as long as the C^1-solution exists. Therefore the solution remains in the region Ω_1.

We are going to discuss the a priori estimate of $r_x(x, t)$, $s_x(x, t)$. Following Slemrod[2,3], we have

$$\theta'=-f\theta^2-A\theta-Ag', \tag{2.1$_1$}$$

$$\varphi^{\backprime}=-f\varphi^2-A\varphi-Ag^{\backprime}, \tag{2.1$_2$}$$

where

$$'=\frac{\partial}{\partial t}+\lambda\frac{\partial}{\partial x}, \quad {}^{\backprime}=\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x},$$

$$\theta=\mu^{\frac{1}{2}}\frac{\partial r}{\partial x}=b^{-1}v^{-\frac{r+1}{4}}\frac{\partial r}{\partial x}, \quad \varphi=\mu^{\frac{1}{2}}\frac{\partial s}{\partial x}=b^{-1}v^{-\frac{r+1}{4}}\frac{\partial s}{\partial x}, \tag{2.2}$$

$$f=-\mu_r\mu^{-\frac{1}{2}}=\frac{r+1}{4}\,bv^{-\frac{3-r}{4}}, \quad g=\int^{r-s}\frac{1}{2}\,\mu^{-\frac{1}{2}}(\xi)d\xi. \tag{2.3}$$

Then

$$g'=\frac{1}{2}\,\mu^{-\frac{1}{2}}(r-s)'=\mu^{-\frac{1}{2}}\Phi'=\mu^{\frac{1}{2}}v'=b^{-1}v^{-\frac{r+1}{4}}\,v', \tag{2.4}$$

$$\left(\frac{1}{f}\right)'=\left(\frac{4}{r+1}\,b^{-1}v^{-\frac{3-r}{4}}\right)=C^{-1}b^{-1}v^{-\frac{r+1}{4}}\,v', \tag{2.5}$$

where

$$C=\frac{r+1}{3-r}. \tag{2.6}$$

(2.4), (2.5) implies

$$g'=\left(\frac{C}{f}\right)'. \tag{2.7}$$

Moreover, we set

$$w=\theta+\frac{AC}{f}, \tag{2.8}$$

so that we can rewrite $(2.1)_1$ as

$$w' = -f\left(W - \frac{AC}{f}\right)^2 - A\left(w - \frac{AC}{f}\right), \tag{2.9$_1$}$$

or

$$w' = -fw^2 + A(2C-1)w + \frac{A^2C}{f}(1-C). \tag{2.9$_2$}$$

Similarly, put $y = \varphi + \frac{AC}{f}$, $(2.1)_2$ implies

$$y' = -fy^2 + A(2C-1)y + \frac{A^2C}{f}(1-C).$$

Lemma 2.2. *Consider the initial value problem*

$$z'(t) = -F(t)z^2(t) + A(2C-1)z(t) + \frac{A^2C}{F(t)}(1-C), \tag{2.10}$$

$$z(0) = z_0, \tag{2.11}$$

where $A>0$, $F(t) \in C^1(R_+)$, $F_* = \inf_{t>0} F(t) > 0$.

(1) *If* $0<C\leqslant 1$, $z_0 \geqslant 0$, *then the initial value problem* (2.10), (2.11) *has the a priori estimate for C^1–solution* $z(t)$:

$$0 \leqslant z(t) \leqslant \max\left\{\frac{AC}{F_*},\ z_0\right\}. \tag{2.12}$$

(2) *If* $1 \leqslant C < \infty$, $z_0 < 0$, *then the initial value problem* (2.10), (2.11) *has a C^1–solution for only a finite time.*

Proof (1) The proof is done by contradiction. If the a priori estimate (2.12) fails, then there exists $\tau > 0$, such that

$$z(\tau) < 0, \tag{2.13}$$

or exists $\tau > 0$, such that

$$z(\tau) > \max\left\{\frac{AC}{F_*},\ z_0\right\}. \tag{2.14}$$

However, from (2.13), since $z_0 \geqslant 0$, there exists $0 < \delta \leqslant \tau$, such that

$$z(t) < 0,\ \tau - \delta < t \leqslant \tau, \tag{2.15}$$

$$z(\tau - \delta) = 0. \tag{2.16}$$

(2.16), (2.10) and $F_* > 0$ imply

$$z'(\tau - \delta) > 0, \text{ as } 0 < C < 1. \tag{2.17}$$

(2.16), (2.17) contradict (2.15). As $C = 1$,

$$z(t) = 0,\ t \geqslant \tau - \delta, \tag{2.18}$$

is the unique solution of the initial value problem (2.10), (2.16). But (2.18) contradicts (2.15) also.

From (2.14), there exists $0 < \delta \leqslant \tau$, such that

$$z(t) > \max\left\{\frac{AC}{F_*},\ z_0\right\},\quad \tau - \delta < t < \tau, \tag{2.19}$$

$$z(\tau - \delta) = \max\left\{\frac{AC}{F_*},\ z_0\right\}. \tag{2.20}$$

Rewrite (2.10) as

$$z'(t) = -F\left(z - \frac{AC}{F}\right)^2 - A\left(z - \frac{AC}{F}\right). \tag{2.21}$$

(2.19), (2.21) imply

$$z'(t) < 0, \ \tau - \delta < t \leqslant \tau. \tag{2.22}$$

(2.20), (2.22) contradict (2.19).

(2) Since $1 \leqslant C < \infty$, $z_0 < 0$, it is easy to prove that $z(t) < 0$ as long as C^1-solution $z(t)$ exists, and

$$z'(t) \leqslant -F_* z^2(t) + A(2C-1)z(t). \tag{2.23}$$

Consider the following initial value problem

$$\begin{cases} y'(t) = -F_* y^2(t) + A(2C-1)y(t), \\ y(0) = z_0. \end{cases} \tag{2.24}$$

Solving (2.24) by elementary method, we get

$$y(t) = \frac{A(2C-1)z_0}{F_* z_0 (1 - e^{-A(2C-1)t}) + A(2C-1)e^{-A(2C-1)t}}.$$

Obviously, $y(t)$ goes to $-\infty$ in a finite time for $z_0 < 0$. Comparing (2.23), (2.24), by comparison theorem, we get $z(t) \leqslant y(t) < 0$ as long as $z(t)$ exists. Therefore $z(t)$ exists for only a finite time.

Lemma 2.3. *Suppose* $6 < \gamma \leqslant 1$. *Under the condition* (A), (C), *any* C^1*-solution to the initial value problem* $(E)_1$, $(I)_1$ *has the a priori estimate*:

$$|r_x(x, t)| \leqslant M, \ |s_x(x, t)| \leqslant M,$$

where M *only depends on* K, r, A *and the bounds of* $r_0(x)$, $s_0(x)$, $r_0'(x)$, $s_0'(x)$.

Proof Following the proof of the part (1) for Lemma 2.2, we can prove that C^1-solution to the equation (2.9) has the a priori estimate

$$0 \leqslant w(x, t) \leqslant \max\left\{\frac{AC}{f_*}, \ w_0\right\}, \tag{2.25}$$

if

$$w(x, 0) \geqslant 0, \ x \in R. \tag{2.26}$$

From conditions (A), (C), $f_* = \inf f(x, t) > 0$, $w_0 = \sup w(x, 0) < \infty$. Since $\mu_r = \mu'(v)v_r$, $\Phi = \frac{1}{2}(r - s)$, $\mu v_r = \frac{1}{2}$, we have $\mu_r = \frac{1}{2}\mu^{-1}\mu' = -\frac{\gamma+1}{4v}$. From (2.7), (2.2), (2.3), (2.6),

$$w = \mu^{\frac{1}{2}}(r_x - AC\mu_r^{-1}) = \mu^{\frac{1}{2}}\left(r_x + \frac{4Av}{3-\gamma}\right).$$

Consequently, condition (2.26) is equivalent to the condition (B). r_x is bounded. Its bound depends on K, γ, A and the bounds of $r_0(x)$, $s_0(x)$, $r_0'(x)$, $s_0'(x)$. Similarly, s_x has an analogues a priori estimate.

The a priori estimates in Lemmas 2.1, 2.3, and the local existence theorem for $(E)_1$, $(I)_1$ give the following global existence theorem.

Theorem 2.4. *Suppose* $0 < \gamma \leqslant 1$. *Under conditions* (A), (C), *if condition* (B) *holds for all* $x \in R$, *then the initial value problem* (E), (I) *has a unique global smooth* *solution.*

According to the part (2) of Lemma 2.2, the following theorem can be proved by contradition.

Theorem 2.5. *Suppose* $1 \leqslant \gamma < 3$. *Under conditions* (A), (C), *if condition* (B) *fails at any point* $x \in R$, *then the initial value problem* (E), (I) *has a* C^1*-solution for only a finite time.*

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