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PROPAGATION OF SINGULARITIES FOR SOLUTION OF SEMILINEAR WAVE EQUATIONS

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Abstract

In order better to research the singularities of the solutions $u\in H^s_{loc}(\Omega)$, $\Omega\subset R^n$, $s>\frac{n}{2}+1$, for semilinear hyperbolic equations $\square\, u=f(u, Du)$, in this paper, a kind of weighted Sobolev space $(H^s)^q_{P_\mu}$, $\mu=1, 2$, $p_1=D_t-|D_x|$, $P_2=D_t+|D_x|$, closely related with the solutions of the equations, is presented. It is discussed thattheir products tacitly keep roughly H^{3s-n} microlocal regularity on the characteristic directions for P_μ and invariance under nonlinear maps. Then it is obtained that roughly H^{3s-n} propagation of singularities theorem is valid for $\square\, u=f(u)$.

§0. Introduction

In 1979, Rauch [1] first analysed regularities of the solutions for semilinear hyperbolic equations $\square u=f(u)$, $\Omega\subset\mathbb{R}^n$, $S>\frac{n}{2}$, f is a polynomial on u. He showed that if the regularity of the solution u at some point in $T^*(\Omega)\backslash 0$ is less than or equal to $H^{2s-\frac{n}{2}+1}$, then it propogates along bicharacteristic of $\square$. In 1981 [5], in 1982 [2], in 1984 [6], [8] a roughly $H^{2s-\frac{n}{2}}$ propagation of singularities theorem for $\square u=f(u, Du)$ is given. In 1982 M. Beals [3] tacitly acquired the result about roughly H^{3s-n} propagation of singularities for $\square u=f(u)$. In 1984 Chen Shuxing also analysed roughly H^{3s-n} regularities for $\square u=f(u)$.

As we know, Rauch [1]. M. Beal and M. Reed [2] showed that if $\frac{n}{2}<s<r<2s-\frac{n}{2}$, $H^s\cap H^r_{m1}(t, x, \tau, \xi)$ constitutes an algebra. In order to acquire roughly H^{3s-n} propagation of singularities theorem for $\square u=f(u, Du)$, it is necessary to prove roughly H^{3s-n} microlocal regularities of the products on the characteristic directions of $\square$ and invariance under nonlinear maps. This is the content in this paper.

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§1. The Microlocal Theorem on Products

Let $\alpha, \beta \geqslant 0$. $\Gamma_1 \subset \mathbb{R}^n$ is a conic neighborhood. Write $P_1 = D_t - |D_x|$, $P_2 = D_t + |D_x|$. Define

$$\|u\|_{(H^r)^\beta_{P_\mu}(\Gamma_1)} = \|\langle(\tau, \xi)\rangle^r \langle P_\mu(\tau, \xi)\rangle^\beta \hat{u}(\tau, \xi)\|_{L^2(\Gamma_1)},$$

where $$\langle(\tau, \xi)\rangle = (1 + |(\tau, \xi)|^2)^{1/2}, \quad |(\tau, \xi)| = \left(\tau^2 + \sum_{i=1}^{n-1} \xi_i^2\right)^{1/2}.$$

$P_\mu(\tau, \xi)$ is the symbol of P_μ, $\mu = 1, 2$.

$$\|n\|_{(H^s)^\alpha_{P_\mu}(\mathbb{R}^n) \cap (H^r)^\beta_{P_\mu}(\Gamma_1)} = \|u\|_{(H^s)^\alpha_{P_\mu}(\mathbb{R}^n)} + \|u\|_{(H^r)^\beta_{P_\mu}(\Gamma_1)},$$

$$(H^s)^\alpha_{P_\mu} = (H^s)^\alpha_{P_\mu}(\mathbb{R}^n) = \{u \in S';\ \|u\|_{(H^s)^\alpha_{P_\mu}(\mathbb{R}^n)} < +\infty\},$$

$$(H^s)^\alpha_{P_\mu}(\Gamma_1) = \{u \in S';\ \|u\|_{(H^s)^\alpha_{P_\mu}(\Gamma_1)} < +\infty\}.$$

Write $$N_1 = \{(\tau, \xi);\ \tau > |\xi|\},$$
$$N_2 = \{(\tau, \xi);\ -\tau > |\xi|\}.$$

Definition 1.1. $u \in (H^s)^\alpha_{P_1} \cap (H^r)^\beta_{P_1}(t_0, x_0, \tau_0, \xi_0)$, $(t_0, x_0, \tau_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash 0)$ *means that there exists a smooth* $\phi(t, x)$, *supported near* (t_0, x_0) *with* $\phi(t_0, x_0) = 1$ *and a cone* Γ *in* $\mathbb{R}^n \backslash 0$ *about the direction* (τ_0, ξ_0) *such that*

(i) $\langle(\tau, \xi)\rangle^s \langle\tau - |\xi|\rangle^\alpha \widehat{\phi u}(\tau, \xi) \in L^2(\mathbb{R}^n)$,

(ii) $\langle(\tau, \xi)\rangle^r \langle\tau - |\xi|\rangle^\beta \widehat{\phi u}(\tau, \xi) \in L^2(\Gamma)$.

In the same way, we can define $(H^s)^\alpha_{P_2} \cap (H^r)^\beta_{P_2}(t_0, x_0, \tau_0, \xi_0)$.

Let $K \subset T^*(\mathbb{R}^n) \backslash 0$ be a closed cone for (τ, ξ). If $u \in (H^s)^\alpha_{P_\mu} \cap (H^r)^\beta_{P_\mu}(t, x, \tau, \xi)$ $\forall (t, x, \tau, \xi) \in K$, we say $u \in (H^s)^\alpha_{P_\mu} \cap (H^r)^\beta_{P_\mu}(K)$. Note when $\alpha = \beta = 0$ in the definition, it is called $H^s \cap H^r_{m1}(t_0, x_0, \tau_0, \xi_0)$ (see [2]). If condition (i) is valid, we say $u \in (H^s)^\alpha_{P_1}(t_0, x_0)$. In this paper we always suppose $\alpha, \beta \geqslant 0$, $0 \leqslant s \leqslant r$, $(t, x) \in \Omega \subset \mathbb{R} \times \mathbb{R}^{n-1}$.

Let g_1, g_2, be nonnegative functions. If there exists a constant $c_1 > 0$ such that $g_1 \leqslant c_1 g_2$ we can write $g_1 \lesssim g_2$. If there also exists a constant $c_1 > 0$ such that $g_2 \leqslant c_2 g_1$, we say $g_1 \sim g_2$.

We often apply the following lemma and do not point it out (see [3] or [4]).

Lemma *Suppose that* K: $R^n \times R^n \to C$ *is a locally integrable mensureable function such that either*

$$\sup_{(\tau, \xi)} \int |K(\tau, \xi, \lambda, \eta)|^2 d\lambda\, d\eta < \infty,$$

or

$$\sup_{(\lambda, \eta)} \int |K(\tau, \xi, \lambda, \eta)|^2 d\tau\, d\xi < \infty$$

holds. Then, the map

$$(g, h) \to \int K(\tau, \xi, \lambda, \eta) g(\tau - \lambda, \xi - \eta) h(\lambda, \eta) d\lambda\, d\eta$$

extends from $\mathscr{D}(\mathbb{R}^n)\times\mathscr{D}(\mathbb{R}^n)$ *to a continuous bilinear map of* $L^2(R^n)\times L^2(R^n)$ *to* $L^2(R^n)$. *That is*

$$\left\|\int K(\tau,\ \xi,\ \lambda,\ \eta)g(\tau-\lambda,\ \xi-\eta)\ h(\lambda,\ \eta)d\lambda\, d\eta\right\|_{L^2}\leqslant c\|g\|_{L^2}\cdot\|h\|_{L^2}.$$

Applying Hölder ineguality, we can easily prove the following often used result.

Lemma 1. 2. *Let* $s_i\geqslant 0,\ \alpha_i\geqslant 0,\ i=1,\ 2.$ *If* $s_1+s_2>\frac{n-1}{2}$ *and* $s_1+s_2+\alpha_1+\alpha_2>\frac{n}{2}$, *integral*

$$\sup_{(\tau,\xi)}\int_{\mathbb{R}\times\mathbb{R}^{n-1}}\frac{d\lambda\, d\eta}{\langle(\lambda,\ \eta)\rangle^{2s_1}\langle\lambda-|\eta|\rangle^{2\alpha_1}\langle(\tau-\lambda,\ \xi-\eta)\rangle^{2s_2}\langle\tau-\lambda-|\xi-\eta|\rangle^{2\alpha_2}}<\infty.$$

If we use $\langle\lambda+|\eta|\rangle$ instead of $\langle\lambda-|\eta|\rangle$ or $\langle\tau-\lambda+|\xi-\eta|\rangle$ instead of $\langle\tau-\lambda-|\xi-\eta|\rangle$, or both instead of them, the integral is valid also.

Write $s=\min\{s_1,\ s_2\},\ r\leqslant\min\{r_1,\ r_2\},\ \alpha=\min\{\alpha_1,\ \alpha_2\},\ \beta=\min\{\beta_1,\ \beta_2\}$. It always means a theorem is valid respectively for $\mu=1$ and $\mu=2$ that P_μ appears in the theorem.

Theorem 1.3. *Let* $u_i\in(H^{s_i})^{\alpha_i}_{P_\mu}\cap(H^{r_i})^{\beta_i}_{P_\mu}(t_0,\ x_0,\ \tau_0,\ \xi_0)$. $i=1,\ 2.$ *If*

$$\begin{cases}\gamma+\beta<\min\{\gamma_1+\beta_1+\gamma_2+\beta_2,\ s_1+\alpha_1+s_2+\alpha_2,\ \gamma_1+\beta_1+s_2+\alpha_2,\ s_1+\alpha_1+\gamma_2+\beta_2\}-n/2,\\ \gamma+\beta<s_1+s_2-(n-1)/2,\ (\tau_0,\ \xi_0)\in N_\mu,\\ \gamma+\beta<\min\{s_1+s_2+\alpha,\ r_1+s_2,\ r_2+s_1\}-(n-1)/2,\ (\tau_0,\ \xi_0)\notin N_\mu,\end{cases}$$

then $u_1u_2\in(H^r)^{\beta}_{P_\mu}(t_0,\ x_0,\ \tau_0,\ \xi_0)$.

Let Γ_1 be a conic neighborhood of $(\tau_0,\ \xi_0)$ such that $u_i\in(H^{r_i})^{\beta_i}_{P_\mu}(\Gamma_1)$. Then there exiss a conic neighborhood $\Gamma\subset\subset\Gamma_1$ of $(\tau_0,\ \xi_0)$ with a constant $c<0$ such that the following inequality is valid

$$\|u_1u_2\|_{(H^r)^{\beta}_{P_\mu}(\Gamma)}\leqslant c\prod_{i=1}^{2}\|u_i\|_{(H^{s_i})^{\alpha_i}_{P_\mu}(\mathbb{R}^n)\cap(H^{r_i})^{\beta_i}_{P_\mu}(\Gamma_1)},$$

$\forall u_i\in(H^{s_i})^{\alpha_i}_{P_\mu}(R^n)\cap(H^{r_i})^{\beta_i}_{P_\mu}(\Gamma_1)$.

Proof We prove it only for $\mu=1$, i. e., $P_1=D_t-|D_x|$, because the proof is the same for P_2. We assume u_i have been multiplied by a smooth cutoff function so as to be compactly supported. We consider for any $(\tau,\ \xi)\in\Gamma$

$$\begin{aligned}I&=\langle(\tau,\ \xi)\rangle^{\gamma}\langle(\tau-|\xi|\rangle^{\beta}\widehat{u_1u_2}(\tau,\ \xi)\\&=\sum_{i=1}^{4}\int_{\Omega_j}K_j g_{1j}(\lambda,\ \eta)g_{2i}(\tau-\lambda,\ \xi-\eta)d\lambda\, d\eta\\&=I_1+I_2+I_3+I_4,\end{aligned}$$

where $g_{1j},\ g_{2j}\in L^2,\ \Omega_j,\ K_j$ are given later on.

(1) In $\Omega_1=\{(\lambda,\ \eta):\ (\lambda,\ \eta)\in\Gamma_1,\ (\tau-\lambda,\ \xi-\eta)\in\Gamma_1\}$,

$$K_1=\frac{\langle(\tau,\ \xi)\rangle^{\gamma}\langle\tau-|\xi|\rangle^{\beta}}{\langle(\lambda,\ \eta)\rangle^{\gamma_1}(\lambda-|\eta|\rangle^{\beta_1}\langle(\tau-\lambda,\ \xi-\eta)\rangle^{\gamma_2}\langle\tau-\lambda-|\xi-\eta|\rangle^{\beta_2}}.$$

(i) If $|(\lambda,\ \eta)|<\frac{1}{2}|(\tau,\ \xi)|,\ |(\tau-\lambda,\ \xi-\eta)|\geqslant\frac{1}{2}|(\tau,\ \xi)|$ and notice

$\langle\tau-|\xi|\rangle\rangle\lesssim\langle\tau-\lambda-|\xi-\eta|\rangle+\langle(\lambda,\eta)\rangle$, it follows that

$$K_1\lesssim\frac{1}{\langle(\lambda,\eta)\rangle^{\gamma_1+\gamma_2-\gamma}\langle\lambda-|\eta|\rangle^{\beta_1-\beta}\langle\tau-\lambda-|\xi-\eta|\rangle^{\beta_2}}+\frac{1}{\langle(\lambda,\eta)\rangle^{\gamma_1+\gamma_2-(\gamma+\beta)}\langle\lambda-|\eta|\rangle^{\beta_1}\langle\tau-\lambda-|\xi-\eta|\rangle^{\beta_2}}.$$

(ii) If $|(\lambda,\eta)|\geqslant\frac{1}{2}|(\tau,\xi)|$, apllying $\langle\tau-|\xi|\rangle\lesssim\langle\lambda-|\eta|\rangle+\langle(\tau-\lambda,\xi-\eta)\rangle$ it follows that

$$K_1\lesssim\frac{1}{\langle\lambda-|\eta|\rangle^{\beta_1-\beta}\langle(\tau-\lambda,\xi-\eta)\rangle^{\gamma_1+\gamma_2-\gamma}\langle\tau-\lambda-|\xi-\tau|\rangle^{\beta_2}}+\frac{1}{\langle\lambda-|\eta|\rangle^{\beta_1}\langle(\tau-\lambda,\xi-\eta)\rangle^{\gamma_1+\gamma_2-(\gamma+\beta)}\langle\tau-\lambda-|\xi-\eta|\rangle^{\beta_2}}.$$

According to Lemma 1.2 and the given conditions, we have $\sup\int_{\Omega_1}K_1^2d\lambda\,d\eta<\infty$, so that

$$\|I_1\|_{L^2(\Gamma)}\leqslant c\|u_1\|_{(H^{\gamma_1})^{\beta_1}_{\frac{1}{2}}(\Gamma_1)}\|u_1\|_{(H^{\gamma_2})^{\beta_2}_{\frac{1}{2}}(\Gamma_1)}.$$

(2) In $\Omega_2=\{(\lambda,\eta)\notin\Gamma_1,\ (\tau-\lambda,\xi-\eta)\notin\Gamma_1\}$.

Notice the following fact [1].

Soppose $\Gamma_1,\Gamma_2,\Gamma_3\subset R^n\backslash 0$ are closed cone, and $\Gamma\cap(\Gamma_2\cup\Gamma_3)=\phi$. Then there exists a constant $c>0$. such that

$$|(\tau-\lambda,\xi-\eta)|^{l_1},\ |(\lambda,\eta)|^{l_2}\geqslant c|(\tau,\xi)|^{l_1+l_2},$$

$\forall(\tau,\xi)\in\Gamma,\ (\tau-\lambda,\xi-\eta)\in\Gamma_2,\ (\lambda,\eta)\in\Gamma_3$, where $l_1, l_2\geqslant 0$.

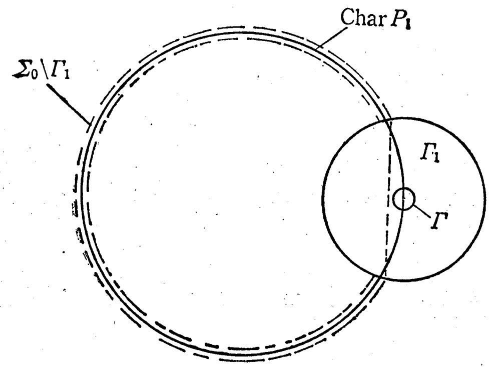

Fig. 1 relation between Char P_1, Γ_1, Γ and $\Sigma_0\backslash\Gamma_1$ in $\tau=1$.

$$K_2=\frac{\langle(\tau,\xi)\rangle^{\gamma}\langle(\tau-|\xi|\rangle^{\beta}}{\left\{\begin{matrix}\langle(\lambda,\eta)\rangle^{s_1}\langle\lambda-|\eta|\rangle^{\alpha_1}\langle(\tau-\lambda,\\ \xi-\eta)\rangle^{s_2}\langle\tau-\lambda-|\xi-\eta|\rangle^{\alpha_2}\end{matrix}\right\}}\lesssim\frac{1}{\left\{\begin{matrix}\langle(\lambda,\eta)\rangle^{s_1+s_2-(\gamma+\beta)}\langle\lambda-|\eta|\rangle^{\alpha_1}\\ \langle\tau-\lambda-|\xi-\eta|\rangle^{\alpha_2}\end{matrix}\right\}}.$$

If $(\tau_0,\xi_0)\in N_1$, it is valid abviosuly, so we suppose $(\tau_0,\xi_0)\notin N_1$ and $(\tau_0,\xi_0)\in\text{Char }P_1=\{(\tau,\xi);\ \tau-|\xi|=0\}$.

Let Σ_0 be an adequately small conic neighborhood of Char P_1. Then we can choose an adequately small conic neighborhood of (τ_0,ξ_0), as Figure 1 shows, such that if (λ,η), $(\tau-\lambda,\xi-\eta)\in\Sigma_0\backslash\Gamma_1$ then $(\tau,\xi)\notin\Gamma$, so that if $(\lambda,\eta)\in\Sigma_0\backslash\Gamma_1$ then $\langle\tau-\lambda-|\xi-\eta|\rangle\sim\langle(\tau-\lambda,\xi-\eta)\rangle$.

$$K_2\lesssim\frac{1}{\langle(\lambda,\eta)\rangle^{s_1+s_2+\alpha_2-(\gamma+\beta)}\langle\lambda-|\eta|\rangle^{\alpha_1}}.$$

Using Lemma 1.2 yields $\sup\int K_2^2d\lambda\,d\eta<\infty$. By analogy, we can analyse it in other case, so that

$$\|I_2\|_{L^2(\Gamma)}\leqslant c\|u\|_{(H^{s_1})^{\alpha_1}_{\frac{1}{2}}(R_n\backslash\Gamma_1)}\cdot\|u_2\|_{(H^{s_2})^{\alpha_2}_{\frac{1}{2}}(R_n\backslash\Gamma_1)}.$$

(3) In $\Omega_3=\{(\lambda,\eta);\ (\lambda,\eta)\in\Gamma_1,\ (\tau-\lambda,\xi-\eta)\notin\Gamma_1\}$.

Notice that suppose Γ_1, Γ_2 are closed cones and $\Gamma\cap\Gamma_2=\phi$, then there exists a constant $c>0$ such that $|(\lambda,\eta)|\geqslant c|(\tau,\xi)|$, $\forall(\tau-\lambda,\xi-\eta)\in\Gamma_2$, $(\tau,\xi)\in\Gamma$.

$$K_3=\frac{\langle(\tau,\xi)\rangle^{\gamma}\langle\tau-|\xi|\rangle^{\beta}}{\langle(\lambda,\eta)\rangle^{\gamma_1}\langle\lambda-|\eta|\rangle^{\beta_1}\langle(\tau-\lambda,\xi-\eta)\rangle^{s_2}\langle\tau-\lambda-|\xi-\eta|\rangle^{\alpha_2}}$$
$$\lesssim\frac{1}{\langle\lambda-|\eta|\rangle^{\beta_1-\beta}\langle(\tau-\lambda,\xi-\eta)\rangle^{\gamma_1+s_2-\gamma}\langle\tau-\lambda-|\xi-\eta|\rangle^{\alpha_2}}$$
$$+\frac{1}{\langle\lambda-|\eta|\rangle^{\beta_1}\langle(\tau-\lambda,\xi-\eta)\rangle^{\gamma_1+s_2-(\gamma+\beta)}\langle\tau-\lambda-|\xi-\eta|\rangle^{\alpha_2}}.$$

Similarly, using $\sup\int K_4^2\,d\lambda\,d\eta<\infty$ it follows that

$$\|I_3\|_{L^2(\Gamma)}\leqslant c\|u_1\|_{(H^{r_1})^{\beta_1}_{P_1}(\Gamma_1)}\cdot\|u_2\|_{(H^{s_2})^{\alpha_2}_{P_1}(\mathbb{R}_n/\Gamma_1)}.$$

(4) In $\Omega_4=\{(\lambda,\eta);\ (\lambda,\eta)\notin\Gamma_1,\ (\tau-\lambda,\xi-\eta)\in\Gamma_1\}$.

Notice that suppose Γ_1, Γ_2 are closed cones and $\Gamma\cap\Gamma_2=\phi$, then there exists a constant $c>0$ such that $|(\tau-\lambda,\xi-\eta)|\geqslant c|(\tau,\xi)|$, $\forall(\tau,\xi)\in\Gamma$, $(\lambda,\eta)\in\Gamma_2$.

$$K_4=\frac{\langle(\tau,\xi)\rangle^{\gamma}\langle\tau-|\xi|\rangle^{\beta}}{\langle(\lambda,\eta)\rangle^{s_1}\langle\lambda-|\eta|\rangle^{\alpha_1}\langle(\tau-\lambda,\xi-\eta)\rangle^{\gamma_2}\langle\tau-\lambda-|\xi-\eta|\rangle^{\beta_2}}$$
$$\lesssim\frac{1}{\langle(\lambda,\eta)\rangle^{s_1+\gamma_2-(\gamma+\beta)}\langle\lambda-|\eta|\rangle^{\alpha_1}\langle\tau-\lambda-|\xi-\eta|\rangle^{\beta_2}}$$
$$+\frac{1}{\langle(\lambda,\eta)\rangle^{s_1+\gamma_2-\gamma}\langle\lambda-|\eta|\rangle^{\alpha_1}\langle\tau-\lambda-|\xi-\eta|\rangle^{\beta_2-\beta}}.$$

Using $\sup\int K_4^2\,d\lambda\,d\eta<\infty$ it follows similarly that

$$\|I_4\|_{L^2(\Gamma)}\leqslant c\|u_1\|_{(H^{s_1})^{\alpha_1}_{P_1}(\mathbb{R}_n\backslash\Gamma_1)}\cdot\|u_2\|_{(H^{r_2})^{\beta_2}_{P_2}(\Gamma_1)}.$$

Adding up previous 4 normal inequalities, the proof is finished.

Corollary 1.4. *Let* $u_i\in(H^{s_i})^{\alpha}_{P_\mu}\cap(H^{r})^{\beta}_{P_\mu}(t_0,x_0,\tau_0,\xi_0)$. *If*

$$\begin{cases} s+\alpha>n/2,\ s-\beta>(n-1)/2,\\ r+\beta<s_1+s_2+2\alpha-n/2,\\ r+\beta<s_1+s_2-(n-1)/2,\ (\tau_0,\xi_0)\in N_\mu,\\ r+\beta<s_1+s_2+\alpha-(n-1)/2,\ (\tau_0,\xi_0)\notin N_\mu, \end{cases}$$

then $u_1u_2\in(H^{r})^{\beta}_{P_\mu}(t_0,x_0,\tau_0,\xi_0)$.

If $s-\alpha>(n-1)/2$ also, $u_1u_2\in(H^{s})^{\alpha}_{P_\mu}\cap(H^{r})^{\beta}_{P_\mu}(t_0,x_0,\tau_0,\xi_0)$.

Remark. If $\alpha=\beta=0$, Corollary 1.4 is Rauch lemma [2].

Theorem 1.5. *If* $s_1\geqslant|s_2|$, $s_1-\alpha>(n-1)/2$ *and* $s_1+\alpha>n/2$, *then*

$$(H^{s_1})^{\alpha_1}_{P_\mu}\cdot(H^{s_2})^{\alpha}_{P_\mu}\subset(H^{s_2})^{\alpha}_{P_\mu}.$$

Proof If $s_2\geqslant0$, using Theorem 1.3 the result is proved immediately. If $s_2<0$, it is easily proved also.

Definition 1.6. *If* $u=v_1+v_2$, $\prod W(v_i)\subset E_i$, *where* $v_i\in(H^{s_i})^{\alpha_i}_{P_i}$, $E_1\equiv\{(\tau,\xi);\ \tau\geqslant0\}$. $E_2\equiv\{(\tau,\xi);\ \tau\leqslant0\}$, $i=1,2$, *we say* $u\in(H^{s_1})^{\alpha_1}_{P_1}\oplus(H^{s_2})^{\alpha_2}_{P_2}$, *and write*

$$\|u\|_{(H^{s_1})^{\alpha_1}_{P_1}\oplus(H^{s_2})^{\alpha_2}_{P_2}}=\operatorname*{Inf}_{u=v_1+v_2}\sum_{i=1}^{2}\|v_i\|_{(H^{s_i})^{\alpha_i}_{P_i}}.$$

One easily proves the following result.

Proposition 1.7. *If* $s_1+\alpha_1=s_2+\alpha_2>\frac{n}{2}$, $s_i-\alpha_j>\frac{n-1}{2}$, $i\neq j$, i, $j=1, 2$, *then* $(H^{s_1})^{\alpha_1}_{P_1}\cdot(H^{s_2})^{\alpha_2}_{P_2}\subset(H^{s_1})^{\alpha_1}_{P_1}\oplus(H^{s_2})^{\alpha_2}_{P_2}$, *and there exists a constant* $c>0$ *such that*

$$\|u_1u_2\|_{(H^{s_1})^{\alpha_1}_{P_1}\oplus(H^{s_2})^{\alpha_2}_{P_2}}\leqslant c\|u_1\|_{(H^{s_1})^{\alpha_1}_{P_1}}\cdot\|u_2\|_{(H^{s_2})^{\alpha_2}_{P_2}}$$

$\forall u_i\in(H^{s_i})^{\alpha_i}_{P_i}$.

Using Theorem 1.5 and Proposition 1.7 we have the following theorem

Theorem 1.8. *If* $s_1+\alpha_1=s_2+\alpha_2>n/2$, $s_i-\alpha_j>(n-1)/2$, i, $j=1, 2$, *then* $(H^{s_1})^{\alpha}_{P_1}\oplus(H^{s_2})^{\alpha}_{P_2}$ *constitutes an algebra.*

Lemma 1.9. *Suppose* $(1, w)\in\operatorname{Char}P_1$, (τ_0, ξ_0) *is not in tangent plane* E *to* Char $\square$ *at* $(1, w)$, Γ *is a conic neighborhood of* (τ_0, ξ_0), *and* $\Gamma\cap E=\phi$. *Then there are conic neighborhoods* Γ^δ *of* $(1, w)$ *and* $\Gamma^{-\delta}=\{(\tau, \xi); -(\tau, \xi)\in\Gamma^\delta\}$ *of* $(-1, -w)$, *such that*

$$\langle(\tau, \xi)\rangle\leqslant c_\Gamma(\langle\tau-\lambda+|\xi-\eta|\rangle+\langle\lambda-|\eta|\rangle)$$

$\forall(\lambda, \eta)\in\Gamma^\delta$ $(\tau-\lambda, \xi-\eta)\in\Gamma^\delta_-$, $(\tau, \xi)\in\Gamma$, *where* $C_\Gamma>0$ *is only dependent of cone* Γ.

Proof Assume without loss of generality that $w=(1, 0, \cdots, 0)\in\mathbb{R}^{n-1}$, so that $E=\{(\tau, \xi); \tau=\xi_1\}$. Then there exists a constant $c_0>0$ such that

$$|\tau-\xi_1|\geqslant 2c_0|(\tau, \xi)|.$$

For an adequately small $\delta>0$, take Γ^δ adequately small.

If $\tau-\xi_1<0$, similarly, one can prove $C_0[(\tau, \xi)|\leqslant-(\tau-\lambda+|\xi-\eta|)-(\lambda-|\eta|)$, so that it has been proved.

Remark. If the locations of (λ, η) and of $(\tau-\lambda, \xi-\eta)$ interchange, similarly, one can acquire

$$\langle(\tau, \xi)\rangle\leqslant C_\Gamma(\langle\tau-\lambda-|\xi-\eta|\rangle+\langle\lambda+|\eta|\rangle).$$

Proposition 1.10. *Suppose* $u_1\in(H^{s_1})^{\alpha_1}_{P_1}\cap(H^r)^{\beta}_{P_1}(t_0, x_0, \tau_0, \xi_0)$, $\tau_0>0$, $u_2\in(H^{s_2})^{\alpha_2}_{P_2}(t_0, x_0)$ *and* $\prod wF(u_2)\subset E_2$, $\alpha=\min\{\alpha_1, \alpha_2\}$. *If*

$$\begin{cases} s_2-\beta>(n-1)/2,\ s_2+\alpha_2>n/2,\\ r+\beta<s_1+s_2+\alpha_1+\alpha_2-n/2,\\ r+\beta<s_1+s_2-(n-1)/2,\ (\tau_0, \xi_0)\notin\operatorname{Char}P_1\cup N_1,\\ r+\beta<s_1+s_2+\alpha-(n-1)/2,\ (\tau_0, \xi_0)\in\operatorname{Char}P_1\cup N_1,\end{cases}$$

then $u_1u_2\in(H^r)^{\beta}_{P_1}(t_0, x_0, \tau_0, \xi_0)$.

Proof The idea of the proof is the same as of Theorem 1.3 and, we adopt the same denotation. The crux of thequestion is in Ω_2 where we easily prove that there exist limited conic neighborhoods Γ^δ_i such that Char $P_1\backslash\Gamma_1\subset\bigcup_i\Gamma^\delta_i$ and exists a constant $c>0$ such that

$$\langle(\tau, \xi)\rangle\leqslant c(\langle\tau-\lambda+|\xi-\eta|\rangle+\langle\lambda-|\eta|\rangle)$$

$\forall(\lambda, \eta)\in\Gamma_i^\delta$, $-(\tau-\lambda, \xi-\eta)\in\Gamma_i^\delta$ and $(\tau, \xi)\in\Gamma$.

It is easy to make limited cones $\Gamma_{ij}^\delta\subset\Gamma_i^\delta$ such that Char $P_1\backslash\Gamma_1\subset\bigcup_{i,j}\Gamma_{ij}^\delta\equiv\Sigma^\delta$, and $\Gamma_{ij}^\delta\cap(\Gamma+\Sigma^\delta\backslash\Gamma_i^\delta)=\phi$ for small conic neighborhood Γ of (τ_0, ξ_0), i. e., if $(\tau_1, \xi_1)\in\Gamma_{ij}^\delta$, then (τ_1, ξ_1) is not in the line which go through some point in Γ and one in $\Sigma^\delta\backslash\Gamma_i^\delta$, as shown in Figure 2. So if $(\lambda, \eta)\in\Gamma_{ij}^\delta$, $-(\tau-\lambda,\xi-\eta)\in\Gamma_i^\delta$, for any $\forall(\tau, \xi)\in\Gamma$ *we have*

$$K_2=\frac{\langle(\tau, \xi)\rangle^\gamma\langle\tau-|\xi|\rangle^\beta}{\langle(\lambda, \eta)\rangle^{s_1}\langle\lambda-|\eta|\rangle^{\alpha_1}\langle(\tau-\lambda, \xi-\eta)\rangle^{s_2}\langle\tau-\lambda+|\xi-\eta|\rangle^{\alpha_2}}$$
$$\lesssim\frac{1}{\langle(\lambda, \eta)\rangle^{s_1+s_2+\alpha-(\gamma+\beta)}\langle\lambda-|\eta|\rangle^{\alpha_1}\langle\tau-\lambda+|\xi-\eta|\rangle^{\alpha_2-\alpha}}$$
$$+\frac{1}{\langle(\lambda, \eta)\rangle^{s_1+s_{12}+\alpha-(\gamma+\beta)}\langle\lambda-|\eta|\rangle^{\alpha_1-\alpha}\langle\tau-\lambda+|\xi-\eta|\rangle^{\alpha_2}}.$$

By using Lemma 1.2 the proof is finished. In other case one can prove it similarly.

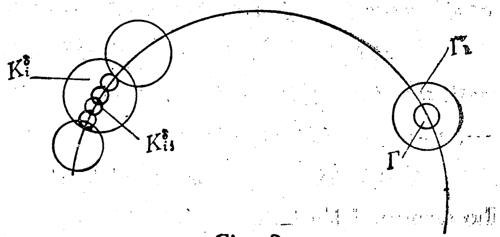

Fig. 2

Remark. If u_1 with u_2, E_1 with E_2, P_1 with P_2, S_1 with S_2, α_1 with α_2 interchange at the same time, then the previous result is valid also.

Definition 1.11. *Suppose* $(t_0, x_0)\in R\times R^{n-1}$. *If there exists* $\varphi\in C_c^\infty(R\times R^{n-1})$, *supported near* (t_0, x_0) *and* $\varphi(t_0, x_0)=1$ *such that* $\varphi u\in(H^{s_2})_{P_1}^{\alpha_1}\oplus(H^{s_2})_{\alpha_2}^{P_2}$, *then we say* $u\in(H^{s_1})_{P_1}^{\alpha_1}\oplus(H^{s_2})_{P_2}^{\alpha_2}(t_0, x_0)$.

By using Corollary 1.4 and Proposition 1.10 it is not difficult to prove the important product theorem we need.

Theorem 1.12 *Suppose* $u_i\in(H^s)_{P_1}^\alpha\oplus(H^s)_{P_2}^\alpha(t_0, x_0)$ *and* $u_i\in(H^r)_{P_\mu}^\beta(t_0, x_0, \tau_0, \xi_0)$, $\tau_0\neq0$. *If*

$$\begin{cases}s-\beta>(n-1)/2,\ s+\alpha>n/2,\ r+\beta>n/2,\\ r+\beta<2s+2\alpha-n/2,\\ r+\beta<2s-(n-1)/2,\ (\tau_0, \xi_0)\notin\text{Char}\,P_\mu,\\ r+\beta<2s+\alpha-(n-1)/2,\ (\tau_0, \xi_0)\in\text{Char}\,P_\mu,\end{cases}$$

then $u_1u_2\in(H^r)_{P_\mu}^\beta(t_0, x_0, \tau_0, \xi_0)$. *If* $s-\alpha>\frac{n-1}{2}$ *also, then*

$$u_1u_2\in(H^s)_{P_1}^\alpha\oplus(H^s)_{P_2}^\alpha(t_0, x_0).$$

§2. Invariance under Nonlinear Maps

Wen can prove the following result.

Proposition 2.1. *Suppose* $s-\beta>(n-1)/2$, $s+\alpha>n/2$, $0\leqslant\sigma\leqslant s$, $0\leqslant\beta\leqslant\alpha$. *Define* $A\in\text{Hom}(L^2(\mathbb{R}^n))$, $A=\langle D\rangle^\sigma\langle D_t-|D_x|\rangle^\beta ia\langle D\rangle^{-\sigma}\langle D_t-|D_x|\rangle^{-\beta}$ *and suppose* $0<\varepsilon_1<\min\{s-(n-1)/2,\ s+\alpha-n/2,\ 1\}$, $0<\varepsilon_2<\min\{s-\beta-(n-1)/2,\ s-\beta+\alpha-n/2,\ 1\}$.

Then there exists a constant $c>0$ *such that for any* $\forall\ \phi\in L^2(\mathbb{R}^n)$ *and any real* $a\in (H^s)^{\alpha}_{P_1}(R^n)$,

$$((A+A^*)\phi,\ \phi)_{L^2}\leqslant c\|\phi\|_{H^{-s_1}}\cdot\|\phi\|_{L^2}\cdot\|a\|_{(H^s)^{\alpha}_{P_1}},\ if\ \beta=0,$$

$$((A+A^*)\phi,\ \phi)_{L^2}\leqslant c\|\phi\|_{(H^0)^{\alpha}_{P_1}}\cdot\|\phi\|_{L^2}\cdot\|a\|_{(H^s)^{\alpha}_{P_1}},\ if\ \beta\neq 0.$$

By using Proposition 2.1 the following result can be acquired.

Lemma 2.2. *Suppose* $s>n/2$, $\alpha<S-(n-1)/2$. *Then there exists a polynomial* P *with* $P(0)=0$ *which is only dependent on* s, α *and* n, *such that*

$$\|e^{ia}-1\|_{(H^s)^{\alpha}_{P_\mu}(R_n)}\leqslant P(\|a\|_{(H^s)^{\alpha}_{P_\mu}(R_n)}).$$

$\forall$ *real* $a\in (H^s)^{\alpha}_{P_\mu}(\mathbb{R}^n)$.

Since $S(\mathbb{R}^n)$ is dense in $(H^s)^{\alpha}_{P_1}(\mathbb{R}^n)$, one can acquire also the following lemma.

Lemma 2.3. *Suppose* $s>(n-1)/2$, $s+\alpha>n/2$. *Then each* $u\in (H^s)^{\alpha}_{P_\mu}(\mathbb{R}^n)$ *is a continuous function which vanishes at infinity.*

Theorem 2.4. *Suppose* f: $C^N\to C$ *or* f: $\mathbb{R}^N\to C$ *is* C^∞ *function with* $f(0, 0, \cdots, 0)=0$. $u_i\in (H^s)^{\alpha}_{P_\mu}(\mathbb{R}^n)$, $i=1, 2, \cdots, N$. *Suppose* $s>n/2$, $s-\alpha>(n-1)/2$. *Then* $f(u_1(\cdot), \cdots, u_N(\cdot))\in (H^s)^{\alpha}_{P_\mu}(\mathbb{R}^n)$.

Proof It can be assumed that u_i is real and f: $R^N\to C$. By Lemma 2.3 it can be assumed that $f\in S(R^N)$. So

$$f(u)=(2\pi)^{-N}\int_{\mathbb{R}^N}(e^{iu\xi}-1)\hat{f}(\xi)d\xi,$$

where $u=(u_1, \cdots, u_N)$. Since

$$e^{iu\xi}-1=\prod_{j=1}^{N}(e^{iu_j\xi_j}-1)+\sum_{J}\prod_{i\in J}(e^{iu_i\xi_i}-1),$$

where the sum is over all subsets of $\{1, \cdots, N\}$ with cardinality $N-1$, by Lemma 2.2 it is true that $\|e^{iu\xi}-1\|_{(H^s)^{\alpha}_{P_\mu}}$ is polynomially bounded. Thus, the integral is absolutely convergent in $(H^s)^{\alpha}_{P_1}$. This proves the result.

Remark. f: $C^N\to C$ need not be analytic.

Definition 2.5. *Suppose* $K\subset R^n$ *is compact set, write*

$$(H^s)^{\alpha}_{P_\mu}[K]\equiv\{u\in\mathscr{D}'(R^n);\ \text{supp}\ u\subset K,\ u\in (H^s)^{\alpha}_{P_\mu}(t, x),\ \forall (t, x)\in K\}.$$

Suppose $\{\phi_i\}$ *is a finite partition of unity, so that* $\phi_i u\in (H_s)^{\alpha}_{P_\mu}(R^n)$ *and* $\Sigma\phi_i=1$ *on a neighborhood of* K. *Let*

$$\|u\|_{(H^s)^{\alpha}_{P}[K]}\equiv\sum_i\|\phi_i u\|_{(H^s)^{\alpha}_{P_\mu}(R_n)}.$$

If $u=u_1+u_2$, $u_i\in (H^s)^{\alpha}_{P_i}[K]$, $i=1, 2$. *then write* $u\in (H^s)^{\alpha}_{P_1}[K]\oplus(H_s)^{\alpha}_{P_2}[K]$. *Define*

$$\|u\|_{(H^s)^{\alpha}_{P_1}[K]\oplus(H^s)^{\alpha}_{P_2}[K]}=\operatorname*{Inf}_{u=u_1+u_2}\sum_{i=1}^{2}\|u_i\|_{[H^s)^{\alpha}_{P_i}[K]}.$$

Proposition 2.6. *Suppose* $K\subset\mathbb{R}^n$ *is a compact set and* $s-\alpha>\frac{n-1}{2}$, $s+\alpha>\frac{n}{2}$. *Then*

(i) $(H^s)^{\alpha}_{P_\mu}[K]$ *is a Banach algebra,*

(ii) $(H^s)^\alpha_{P_1}[K]\oplus(H^s)^\alpha_{P_2}[K]$ *is a Banach algebra.*

Proof It can be so proved as Theorem 2.4.

Theorem 2.7. *Suppose* $s-\alpha>(n-1)/2$, $s>n/2$. *Let* $f: C^N\to C$ *or* $f: R^N\to C$ *be* C^∞, $u_j\in(H^s)^\alpha_{P_1}\oplus(H^s)^\alpha_{P_2}(t, x)$, $j=1, 2, \cdots, N$, $(t, x)\in\Omega\subset R^n$. *Then*

$$f(u_1, \cdots, u_N)\in(H^s)^\alpha_{P_1}\oplus(H^s)^\alpha_{P_2}(t, x)$$

and the chain rules

$$\frac{\partial}{\partial x_i}f(u)=\sum_{j=1}^N\frac{\partial f(u)}{\partial u_j}\cdot\frac{\partial u_j}{\partial x_i}$$

is valid as an identity in $(H^{s-1})^\alpha_{P_1}\oplus(H^{s-1})^\alpha_{P_2}(t, x)$.

Commutator Lemma 2.8. *Suppose* $b_0(\tau, \xi)\in S^0_{1,0}$, $a(t, x)\in(H^{s_1})^\alpha_{P_\mu}\cap(H^{r+\varepsilon})^\beta_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$, $v(t, x)\in(H^{s_2})^\alpha_{P_\mu}\cap(H^r)^\beta_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$. *If*

$$\begin{cases} s_1-\varepsilon-\beta>(n-1)/2,\ s_1-\varepsilon+\alpha>n/2,\ 0\leqslant\varepsilon\leqslant1, \\ s_2-\beta>(n-1)/2,\ s_2+\beta>n/2, \\ \gamma+\varepsilon+\beta<s_1+s_2+2\alpha-n/2, \\ \gamma+\varepsilon+\beta<s_1+s_2-(n-1)/2,\ (\zeta_0, \xi_0)\in N_\mu, \\ \gamma+\varepsilon+\beta<s_1+s_2+\alpha-(n-1)/2,\ (\zeta_0, \xi_0)\notin N_\mu, \end{cases}$$

then $[b_0(D), a(t, x)]v\in(H^{\gamma+\varepsilon})^\beta_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$, *and for the conic neighborhoods* $\Gamma\subset\subset\Gamma_1$ *of* (τ_0, ξ_0),

$$\|[b_0, a]v\|_{(H^{r+\varepsilon})^\beta_{P_\mu}(\Gamma)}\leqslant c\|a\|_{(H^{s_1})^\alpha_{P_\mu}(\mathbb{R}_n)\cap(H^{r+\varepsilon})^\beta_{P_\mu}(\Gamma_1)}\cdot\|v\|_{(H^{s_2})^\alpha_{P_\mu}(\mathbb{R}_n)\cap(H^r)^\alpha_{P_\mu}(\Gamma_1)}$$

$\forall a\in(H^{s_1})^\alpha_{P_\mu}(\mathbb{R}^n)\cap(H^{r+\varepsilon})^\beta_{P_\mu}(\Gamma_1)$, $v\in(H^{s_2})^\alpha_{P_\mu}(\mathbb{R}^n)\cap(H^r)^\beta_{P_\mu}(\Gamma_1)$.

Proof It can be so proved as Theorem 1.3, only except using in Ω_4 the fact that if $|(\lambda, \eta)|<\frac{1}{2}|(\tau, \xi)|$ then $|b_0(\tau, \xi)-b_0(\tau-\lambda, \xi-\eta)|\lesssim\frac{\langle(\lambda, \eta)\rangle}{\langle(\tau, \xi)\rangle}$.

Proposition 2.9. *Suppose cone* $\Gamma\subset\mathbb{R}^n$ *is independent of* t_1, $r>n/2$, $r-\beta>(n-1)/2$, $s_1-\alpha>(n-1)/2$, $s_1+\alpha>n/2$, $\sigma+s_1\geqslant0$. *If*

(i) $\Pi wF(w)\subset\subset\Gamma$, Π *denotes the projection in the dual place,*

(ii) *real function* $a\in(H^{s_1})^\alpha_{P_\mu}\cap(H^r)^\beta_{P_\mu}(\Gamma)$,

(iii) $g\in H^{s_2}\cap(H^\sigma)^\beta_{P_\mu}(\Gamma)$,

(iv) $w\in(H^\sigma)^\beta_{P_\mu}$, *if* $t_1=0$,

and $\dot{w}(t_1)=iaw(t_1)+g$, *then* $w\in(H^\sigma)^\beta_{P_\mu}$.

Proof By the calculation of pseudodifferential operators, the energy inequality

$$\frac{d}{dt_1}\|w\|_{(H^\sigma)^\beta_{P_\mu}}\leqslant c_1\|w\|_{(H^\sigma)^\beta_{P_\mu}}+c_2$$

can be acquired, so that by Gronwell inequality the result is proved.

Proposition 2.10. *Suppose* $\nu(t_1)=e^{iat_1}-1$, $a\in(H^s)^\alpha_{P_\mu}\cap(H^r)^\beta_{P_\mu}$ $(t_0, x_0, \tau_0, \xi_0)$.

If

$$\begin{cases} s>n/2,\ s-\beta>(n-1)/2,\ s-\alpha>(n-1)/2, \\ r+\beta<2s+2\alpha-n/2, \\ r+\beta<2s-(n-1)/2,\ (\tau_0,\ \xi_0)\in N_\mu, \\ r+\beta<2s+\alpha-(n-1)/2,\ (\tau_0,\ \xi_0)\notin N_\mu, \end{cases}$$

then $v(t_1)\in(H^s)^\alpha_{P_\mu}\cap(H^r)^\beta_{P_\mu}(t_0,\ x_0,\ \tau_0,\ \xi_0)$.

Proof The equation $\dot v(t_1)=iav(t_1)+ia,\ v(0)=0$ is reduced to a proper form. Then after using Proposition 2.9 time and again, to make bootstrap arguments is sufficient.

Theorem 2.11. *Suppose* f: $C^N\to C$ *or* f: $\mathbb{R}^N\to C$ *is* C^∞ *function and* $u_j\in(H^s)^\alpha_{P_\mu}\cap(H^r)^\beta_{P_\mu}(t_0,\ x_0,\ \tau_0,\ \xi_0)$, $j=1,\ 2,\ \cdots,\ N$. *If*

$$\begin{cases} s-\beta>(n-1)/2,\ s-\alpha>(n-1)/2,\ s>n/2, \\ \gamma+\beta<2s+2\alpha-n/2, \\ \gamma+\beta<2s-(n-1)/2,\ (\tau_0,\ \xi_0)\in N_\mu, \\ \gamma+\beta<2s+\alpha-(n-1)/2,\ (\tau_0,\ \xi_0)\notin N_\mu, \end{cases}$$

then $f(u_1,\ \cdots,\ u_N)\in(H^s)^\alpha_{P_\mu}\cap(H^r)^\beta_{P_\mu}(t_0,\ x_0,\ \tau_0,\ \xi_0)$.

Proof First of all, one can prove a result which is analogous with Lemma 2.2. Then apply the argument similar to that of Theorem 2.4.

Theorem 2.12. *Suppose* f $:C^N\to C$ *or* f: $\mathbb{R}^N\to C$ *is* C^∞ *function,* $u_j\in(H^s)^\alpha_{P_1}\oplus(H^s)^\alpha_{P_2}(t,\ x)$ *and* $u_j\in(H^r)^\beta_{P_\mu}(t_0,\ x_0,\ \tau_0,\ \xi_0)$, $j=1,\ 2,\ \cdots,\ N$. *If*

$$\begin{cases} s-\beta>(n-1)/2,\ s-\alpha>(n-1)/2,\ s>n/2, \\ r+\beta<2s+2\alpha-n/2, \\ r+\beta<2s-(n-1)/2,\ (\tau_0,\ \xi_0)\notin \mathrm{Char}\ P_\mu, \\ r+\beta<2s+\alpha-(n-1)/2,\ (\tau_0,\ \xi_0)\in \mathrm{Char}\ P_\mu, \end{cases}$$

then

$$f(u_1,\ \cdots,\ u_N)\in(H^s)^\alpha_{P_1}\oplus(H^s)^\alpha_{P_2}(t_0,\ x_0),$$
$$f(u_1,\ \cdots,\ u_N)\in(H^r)^\beta_{P_\mu}(t_0,\ x_0,\ \tau_0,\ \xi_0).$$

Proof One can prove several results which are respectively analogous to Commutator Lemma 2.8, Proporsition 2.10, Lemma 2.2, and then apply the argument method in Theorem 2.4.

§3. Singularities Propagation of Solutions

Theorem 3.1. *Suppose* $u\in H^s_{loc}(\Omega)$, $s>\frac{n}{2}$ *is a distribution solution in* $\Omega\subset R^n$ *for* $\Box u=f(u)$, *where* f *is* C^∞ *with respect to* u. *Then* $u\in(H^s)^\alpha_{P_1}\oplus(H^s)^\alpha_{P_2}(t,\ x)$, $\forall(t,\ x)\in\Omega$, *and* $\forall\alpha<s-\frac{n-1}{2}+1$.

Proof $u\in(H^s)^0_{P_1}\oplus(H^s)^0_{P_2}(t, x)=H^s(t, x)$. If for $\alpha<s-\frac{n-1}{2}$ we have

$$u\in(H^s)^{\alpha}_{P_1}\oplus(H^s)^{\alpha-1}_{P_2}(t, x),\ (t, x)\in\dot{\Omega},$$

we can prove for $0\leqslant\alpha<s-\frac{n-1}{2}$ $u\in(H^s)^{\alpha+1}_{P_1}\oplus(H^s)^{\alpha+1}_{P_2}(t, x)$.

Theorem 3.2. *Suppose $u\in H^s_{loc}(\Omega)$, $s>\frac{n}{2}$ is a distribution solution in $\Omega\subset R^n$ for $\Box u=f(u)$, where f is C^∞ for u, ν is a null-bicharacteristic of $\Box$ passing through $(t_0, x_0, \tau_0, \xi_0)$. If $u\in H^r(t_0, x_0, \tau_0, \xi_0)$, and $r<3s-n+2$, then $u\in H^r(\nu)$.*

Proof By Hörmander theorem on singularity propagation, Theorem 3.1 and Theorem 2.12, it is easy to prove this theorem.

It can be proved by using pseudodifferential operators with nonsmooth coefficients that roughly H^{3s-n} propagation theorem of singularities for $\Box u=f(u, Du)$ is valid also.

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