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ENUMERATION OF ROOTED VERTEX NON-SEPARABLE PLANAR MAPS

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Abstract

In a rooted planar map, the rooted vertex is said to be non-separable if the vertex on the boundary of the outer face as an induced graph is not a cut-vertex.

In this paper, the author derives a functional equation satisfied by the enumerating function of rooted vertex non-separable planar maps dependent on the edge number and the number of the edges on the outer face boundary, finds a parametric expression of its solution, and obtains an explicit formula for the function.

Particularly, the number of rooted vertex non-separable maps only replying on the edge number and that of rooted vertex non-separable tree-like maps defined in [4] according to the two indices, the edge number and the number of the edges on the outer face boundary, or only one index, the edge number, are also determined.

§ 1. Introduction

General planar maps may be divided into two classes: rooted vertex separable maps and rooted vertex non-separable maps. Of course, their meanings are just what the terms suggest. However, we should mention here that the rooted vertex is separable or non-separable according as so is it on the boundary of the outer face as an induced graph.

As for general rooted planar maps, the number $H_{m,n}$ of rooted $[m, n]$-maps, i. e., maps with m edges and the boundary of the outer face having n edges, has been found in [4]

$$H_{m,n}=\sum_{s=\lceil n/2\rceil+1}^{\min(m+1,n+2)} 3^{m-s+1}T(n, s)\frac{s}{n+1}\binom{2m-s+1}{m}, \tag{1.1}$$

where

$$T(n, s)=\begin{cases}(-1)^{s+1}(U_{s-1}(n)+U_{s-2}(n)+(-1)^{n+1}Q_s(n)), & \lceil n/2\rceil+1\leqslant s\leqslant n+1;\\ (-1)^{n+1}U_n(n)+Q_{n+2}(n), & s=n+2,\end{cases} \tag{1.2}$$

for $n\geqslant 3$,

$$\begin{cases}T(0, 1)=2,\ T(0, 2)=4;\\ T(1, 1)=0,\ T(1, 2)=2,\ T(1, 3)=8;\\ T(2, 2)=2,\ T(2, 3)=-8,\ T(2, 4)=0,\end{cases} \tag{1.3}$$

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$$U_l(n)=\begin{cases}(-2)^l, & l=\lceil n/2\rceil-1;\\ U_l(n-1)+(-1)^{n-2}Q_l(n-2), & \lceil n/2\rceil\leqslant l\leqslant n-1;\\ (-1)^{n-2}Q_n(n-2), & l=n,\end{cases} \tag{1.4}$$

for $\lceil n/2\rceil-1\leqslant l\leqslant n$, $n\geqslant 3$,

$$U_1(3)=-2,\ U_2(3)=0,\ U_3(3)=8, \tag{1.5}$$

and

$$Q_l(i)=\begin{cases}\sum\limits_{j=i-l+2}^{\lceil i/2\rceil+1}(-1)^j\dfrac{2^{l-i+j-1}}{2j-1}\dbinom{i+1}{j-1,\ i-2j+2,\ i-l+2}, & i+1-\lfloor i/2\rfloor\leqslant l\leqslant i+1;\\ \sum\limits_{j=1}^{\lceil i/2\rceil+1}(-1)^j\dfrac{2^j}{2j-1}\dbinom{i+1}{j-1,\ j}, & l=i+2,\end{cases} \tag{1.6}$$

where

$$\binom{n}{n_1,\ n_2,\ \cdots,\ n_k}=\frac{n!}{n_1!n_2!\cdots n_k!(n-n_1-\cdots-n_k)!},\ k\geqslant 1, \tag{1.7}$$

and, $\lfloor x\rfloor$, $\lceil x\rceil$ represents the maximum, minimum integer not greater, less than x respectively.

Let $p(x, y)$, $p^N(x, y)$ and $p^S(x, y)$ be the enumerating function of general rooted planar maps, rooted vertex non-separable and rooted vertex separable maps with the edge number and the number of the edges on the boundary of the outer face as indices respectively. It is easy to see that

$$p(x, y)=1+p^N(x, y)+p^S(x, y), \tag{1.8}$$

where the constant 1 means that the vertex map is distinct from other rooted vertex non-separable maps.

Furthermore, it is easily shown that

$$p(x, g)=(1-p^N(x, y))^{-1}, \tag{1.9}$$

from which and the functional equation of $p(x, y)$ in [4], the functional equation satisfied by $p^N(x, y)$ may be derived as follows

$$\begin{aligned}&((y-1)\alpha(x)-xy)(p^N(x, y))^2-((xy^2+y-1)\alpha(x)-2xy)p^N(x, y)\\&+xy^3\alpha(x)-xy=0,\end{aligned} \tag{1.10}$$

where

$$\alpha(x)=1-p^N(x, 1). \tag{1.11}$$

For convenience, in what follows, we write

$$h^N(x)=p^N(x, 1). \tag{1.12}$$

The main purpose of this paper is to solve the functional equation and find an explicit expression of $H^N_{m,n}$, the number of rooted vertex non-separable planar maps with m edges and the outer face boundary having n edges, independent of $H_{m,n}$ while H^N_m, the number of rooted vertex non-separable maps with m edges, is determined. Consequently, the corresponding numbers of rooted vertex separable planar maps may be obtained directly.

§ 2. Parametric Expressions

Since the discriminant of (1.10) is

$$\lambda(x, y) = ((xy^2+y-1)\alpha(x)-2xy)^2-4((y-1)\alpha(x)-xy)(xy^3\alpha(x)-xy)$$
$$=(\alpha(x))^2\left[1-2y+(1-2x)y^2+\left(6x-\frac{4x^2}{\alpha(x)}\right)y^3+\left(x^2-4x+\frac{4x^2}{\alpha(x)}\right)y^4\right], \tag{2.1}$$

and we write

$$\eta(x)=x^2-4x+\frac{4x^2}{\alpha(x)}, \tag{2.2}$$

we have

$$\lambda(x, y)=(\alpha(x))^2(1-2y+(1-2x)y^2+(x^2+2x-\eta(x))y^3+\eta(x)y^4). \tag{2.3}$$

However, the functional equation (1.10) may be written as

$$((2(y-1)\alpha(x)-xy)p^N(x, y)-((xy^2+y-1)\alpha(x)-2xy))^2=\lambda(x, y). \tag{2.4}$$

If $\xi(x)$ is a power series of x such that the perfect square in (2.4) becomes zero when $y=\xi(x)$, then $\lambda(x, y)$ has the perfect square factor $(y-\xi)^2$, i. e., the following two equations are satisfied simultaneously:

$$\begin{cases}\lambda(x, \xi)=0;\\ \dfrac{\partial}{\partial y}\lambda(x, y)\Big|_{y=\xi}=0.\end{cases} \tag{2.5}$$

According to the first one of (2.5), we have

$$1-2\xi+(1-2x)\xi^2+(x^2+2x-\eta(x))\xi^3+\eta(x)\xi^4=0. \tag{2.6}$$

The second one of (2.5) means that

$$-2+2(1-2x)\xi+3(x^2+2x-\eta(x))\xi^2+4\eta(x)\xi^3=0. \tag{2.7}$$

After rearrangement and reduction, the following forms may be obtained

$$(1-\xi)^2-2\xi^2(1-\xi)x+\xi^3x^2-\xi^3(1-\xi)\eta(x)=0; \tag{2.8}$$
$$2(1-\xi)+2\xi(2-3\xi)x-3\xi^2x^2+\xi^2(3-4\xi)\eta(x)=0. \tag{2.9}$$

From (2.8) and (2.9), eliminating the terms with $\eta(x)$, we have

$$(3-2\xi)(1-\xi)^2-2\xi^2(1-\xi)^2x-\xi^4x^2=0. \tag{2.10}$$

The discriminant of (2.10) is

$$4\xi^4(1-\xi)^4+4(1-\xi)^2\xi^4(3-2\xi)=4\xi^4(1-\xi)^2(2-\xi)^2. \tag{2.11}$$

Therefore

$$x=-\frac{1}{2\xi^4}(2\xi^2(1-\xi)^2\pm 2\xi^2(1-\xi)(2-\xi))$$
$$=-\frac{(1-\xi)}{\xi^2}(3-2\xi), \text{ or } \frac{1-\xi}{\xi^2}. \tag{2.12}$$

Since $h^N(x)=1-\alpha(x)$ must be a power series of x with all the coefficients non-negative, the only choice of x is the former, i. e.,

$$x=-\frac{(1-\xi)(3-2\xi)}{\xi^2}. \tag{2.13}$$

In this case, substituting (2.13) into (2.8) we derive

$$\eta(x)=\frac{(1-\xi)(9-5\xi)}{\xi^4}. \tag{2.14}$$

On account of (2.2) and (1.11), we find the following parametric expressions

$$\alpha(x)=\frac{(3-2\xi)^2}{\xi(4-3\xi)}. \tag{2.15}$$

$$h^N(x)=-\frac{(1-\xi)(9-7\xi)}{\xi(4-3\xi)}. \tag{2.16}$$

§ 3. Solution of the Functional Equation

Substituting (2.14), (2.15) and (2.16) into (2.3), we may obtain that

$$\lambda(x,\ y)=\frac{(3-2\xi)^4}{\xi^2(4-3\xi)^2}\left[1-\frac{y}{\xi}\right]^2\left[1+\frac{2(1-\xi)}{\xi}y+\frac{(9-5\xi)(1-\xi)}{\xi^2}y^2\right]. \tag{3.1}$$

Let us introduce a new substitution of the parameters,

$$t=\frac{1-\xi}{\xi},\ \text{i. e.,}\ \xi=\frac{1}{t+1}. \tag{3.2}$$

Then, we have

$$\lambda(x,\ y)=\frac{(3t+1)^4}{(4t+1)^2}(1-(t+1)y)^2(1+ty)^2\left[1+\frac{4t(2t+1)y^2}{(1+ty)^2}\right]. \tag{3.3}$$

According to (1.10), the following parametric expression of $p^N(x,\ y)$ may be found

$$2((y-1)\alpha(x)-xy)p^N(x,\ y)=(xy^2+y-1)\alpha(x)-2xy$$
$$\pm\frac{(3t+1)^2}{(4t+1)}(1-(t+1)y)(1+ty)\left[1+\frac{4t(2t+1)y^2}{(1+ty)^2}\right]^{1/2}. \tag{3.4}$$

In order to have $h^N(x)=p^N(x,\ 1)$ just be what we determined before, only the positive sign is available.

Conidering the substitution (3.2), from (2.13) and (2.15), we have

$$x=-t(3t+1), \tag{3.5}$$

$$\alpha=\frac{(3t+1)^2}{4t+1}. \tag{3.6}$$

By substituting (3.5) and (3.6) in (3.4),

$$2((3t+1)(2t+1)^2y-(3t+1)^2)p^N(x,\ y)$$
$$=-t(3t+1)^3y^2+(3t+1)(8t^2+5t+1)y-(3t+1)^2$$
$$+(3t+1)^2(1-(t+1)y)(1+ty)\left[1+\frac{4t(2t+1)y^2}{(1+ty)^2}\right]^{1/2}. \tag{3.7}$$

After dividing both sides of (3.7) by $-(3t+1)^2$, we may obtain

$$2\left[1-\frac{(2t+1)^2}{3t+1}y\right]p^N(x,y)=t(3t+1)y^2-\frac{8t^2+5t+1}{3t+1}y+1$$
$$-(1-(t+1)y)(1+ty)\left[1+\frac{4t(2t+1)y^2}{(1+ty)^2}\right]^{1/2}. \quad (3.8)$$

Let us write

$$z=\frac{t(2t+1)}{(1+ty)^2}y^2. \quad (3.9)$$

Then expanding the root form in (3.8) into Taylor series, we have

$$\left[1+\frac{4t(2t+1)y^2}{(1+ty)^2}\right]^{1/2}=1+\sum_{i=1}^{\infty}(-1)^{i-1}\frac{2(2i-2)!}{(i-1)!i!}z^i. \quad (3.10)$$

In addition, by binomial Theorem,

$$(1+ty)^{-2i}=\sum_{j=0}^{\infty}(-1)^j\binom{2i+j-1}{j}t^jy^j. \quad (3.11)$$

Thus, it can be seen that

$$z^i=\sum_{j=0}^{\infty}(-1)^jt^{i+j}(2t+1)^i\binom{2i+j-1}{j}y^{2i+j}$$
$$=(2t+1)^i\sum_{j=0}^{\infty}(-1)^j\binom{2i+j-1}{j}t^{i+j}y^{2i+j}. \quad (3.12)$$

From (3.10), (3.11) and (3.12), we have

$$\left[1+\frac{4t(2t+1)y^2}{(1+ty)^2}\right]^{1/2}=1+\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j-1}\frac{2}{i}\binom{2i-2}{i-1}\binom{2i+j-1}{j}$$
$$\times(2t+1)^it^{i+j}y^{2i+j}.$$

Let $k=2i+j$ be substituted for j. Then

$$\left[1+\frac{4t(2t+1)y^2}{(1+ty)^2}\right]^{1/2}=1+\sum_{k=2}^{\infty}\left[\sum_{i=1}^{[k/2]}(-1)^{i+k-1}\frac{2}{i}\binom{2i-2}{i-1}\binom{k-1}{k-2i}\right.$$
$$\left.\times(2t+1)^it^{k-i}\right]y^k. \quad (3.13)$$

For convenience, let

$$A_l(t)=\sum_{i=1}^{[l/2]}(-1)^{i+l}\frac{1}{i}\binom{2i-2}{i-1}\binom{l-1}{l-2i}(2t+1)^it^{l-i}. \quad (3.14)$$

Substituting (3.13) into (3.8), after rearranging it, we may find

$$\left[1-\frac{(2t+1)^2}{3t+1}y\right]p^N(x,y)=-\frac{t(4t+1)}{3t+1}y+(t(2t+1)+A_2(t))y^2$$
$$+(A_3(t)-A_2(t))y^3+\sum_{k=4}^{\infty}(A_k(t)-A_{k-1}(t)-t(t+1)A_{k-2}(t))y^k. \quad (3.15)$$

Multiplying both sides of (3.15) by the power series expansion of $\left[1-\frac{(2t+1)^2}{3t+1}\right]^{-1}$, we may obtain

$$p^N(x,y)=\sum_{k=1}^{\infty}\sum_{l=1}^{k-1}B_{l+1}(t)\left[\frac{(2t+1)^2}{3t+1}\right]^{k-l-1}y^k, \quad (3.16)$$

where

$$B_k(t)=\begin{cases}-t(4t+1)(3t+1)^{-1}, & k=1;\\ 0, & k=2;\\ t(2t+1)^2, & k=3;\\ A_k(t)-A_{k-1}(t)-t(t+1)A_{k-2}(t), & k\geqslant 4,\end{cases} \tag{3.17}$$

and $A_l(t)$, $l\geqslant 2$, are determined by (3.14).

§ 4. The Determination of $h^N(x)$

In this section, we calculate the power series of $h^N(x)$ according to its parametric expression obtained before. From (3.5), (1.11) and (3.6), we have

$$h^N(x)=-t\,\frac{9t+2}{4t+1}. \tag{4.1}$$

Thus it can be seen that

$$\frac{d}{dt}h^N=-\frac{2(18t^2+9t+1)}{(4t+1)^2}. \tag{4.2}$$

Moreover, the following derivatives may be found:

$$\frac{d^{m-1}}{dt^{m-1}}(-1)^m(3t+1)^{-m}(4t+1)^{-2}\Big|_{t=0}=-\sum_{i=0}^{m-1}4^{m-1}\left(\frac{3}{4}\right)^i\frac{m-i}{m+i}\binom{m+i}{i}m!, \tag{4.3}$$

$$\frac{d^{m-1}}{dt^{m-1}}(-1)^m(3t+1)^{-m}t(4t+1)^{-2}\Big|_{t=0}=\sum_{i=0}^{m-2}4^{m-2}\left(\frac{3}{4}\right)^i\frac{m-i-1}{m+i}\binom{m+i}{i}m!, \tag{4.4}$$

$$\frac{d^{m-1}}{dt^{m-1}}(-1)^m(3t+1)^{-m}t^2(4t+1)^{-2}\Big|_{t=0}=-\sum_{i=0}^{m-3}4^{m-3}\left(\frac{3}{4}\right)^i\frac{m-i-2}{m+i}\binom{m+i}{i}m!. \tag{4.5}$$

In consequence, we may easily calculate that

$$\begin{aligned}\frac{d^{m-1}}{dt^{m-1}}(-1)^m(3t+1)^{-m}\frac{d}{dt}h^N=&2\Big[18\cdot\sum_{i=0}^{m-3}4^{m-3}\left(\frac{3}{4}\right)^i\frac{m-i-2}{m+i}\binom{m+i}{i}m!\\&-9\sum_{i=0}^{m-2}4^{m-2}\left(\frac{3}{4}\right)^i\frac{m-i-1}{m+i}\binom{m+i}{i}m!+\sum_{i=0}^{m-1}4^{m-1}\left(\frac{3}{4}\right)^i\frac{m-i}{m+i}\binom{m+i}{i}m!\Big]\\=&\begin{cases}2, & m=1;\\ 10\cdot 3^{m-2}\dfrac{(2m-3)!}{(m-2)!}-\displaystyle\sum_{i=0}^{m-3}4^{m-2}\left(\frac{3}{4}\right)^i\frac{m-i}{m+i}\binom{m+i}{i}m!, & m\geqslant 2.\end{cases}\end{aligned} \tag{4.6}$$

By using Lagrange's series expansion Theorem, the following is obtained:

$$h^N(x)=2x+\sum_{m=2}^{\infty}\left[10\cdot 3^{m-2}\frac{(2m-3)!}{(m-2)!\,m!}-\sum_{i=0}^{m-3}4^{m-2}\left(\frac{3}{4}\right)^i\frac{m-i}{m+i}\binom{m+i}{i}\right]x^m. \tag{4.7}$$

The first few terms are

$$h^N(x)=2x+5x^2+26x^3+173x^4+1310x^5+\cdots. \tag{4.8}$$

§ 5. Rooted Vertex Non-Separable Tree-Like Maps

In this section, we investigate the number of combinatorially distinct rooted vertex non-separable tree-like maps defined as in [4], i. e., such maps that would be trees if all the multi-edges were considered as single ones.

Let us denote the enumerating function of rooted vertex non-separable tree-like $[m, 2n]$-maps by $T^N(x, y)$. Then, it is easily shown that

$$T(x, y) = (1 - T^N(x, y))^{-1}, \tag{5.1}$$

where $T(x, y)$ is the enumerating function of rooted tree-like $[m, 2n]$-maps, and notice that $T^N(0, 0) = 0$, i.e., the vertex map is not considered.

From the functional equation of $T(x, y)$ in [4], we may derive that $T^N(x, y)$ satisfies the following functional equation

$$(1-x)(T^N(x, y))^2 - (1-x)T^N(x, y) + xy^2 = 0. \tag{5.2}$$

The discriminant of (5.2) is

$$\lambda(x, y) = (1-x)^2 - 4(1-x)xy^2 = (1-x)^2\left[1 - \frac{4xy^2}{1-x}\right]. \tag{5.3}$$

Thus, it can be obtained that

$$\begin{aligned} T^N(x, y) &= \frac{1}{2(1-x)}\left[1 - x \pm (1-x)\left(1 - \frac{4xy^2}{1-x}\right)^{1/2}\right] \\ &= \frac{1}{2}\left[1 \pm \left(1 - \frac{4xy^2}{1-x}\right)^{1/2}\right]. \end{aligned}$$

Since $T^N(0, 0) = 0$, only the negative sign is available. Therefore, we have

$$T^N(x, y) = \frac{1}{2}\left[1 - \left(1 - \frac{4xy^2}{1-x}\right)^{1/2}\right]. \tag{5.4}$$

Let $z = \frac{xy^2}{1-x}$. By expanding $(1-4z)^{1/2}$ into Taylor series about $z=0$, i. e., $x=0$, $y=0$,

$$(1-4z)^{1/2} = 1 - \sum_{n=1}^{\infty} \frac{2(2n-2)!}{(n-1)!n!} z^n. \tag{5.5}$$

Hence, (5.4) becomes

$$\begin{aligned} T^N(x, y) &= \sum_{n=1}^{\infty} \frac{(2n-2)!}{(n-1)!n!} z^n \\ &= \sum_{n=1}^{\infty} \frac{(2n-2)!}{(n-1)!n!}\left[\frac{xy^2}{1-x}\right]^n. \end{aligned} \tag{5.6}$$

On account of Taylor expansion or binomial Theorem, we may obtain

$$T^N(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{m} \frac{(2n-2)!(m-1)!}{((n-1)!)^2(m-n)!n!} x^m y^{2n}. \tag{5.7}$$

Let $t^N(x)$ denote the enumerating function of rooted vertex non-separable plane trees with the edge number as an index, we have

$$t^N(x) = \sum_{m=1}^{\infty} \frac{(2m-2)!}{(m-1)!m!} x^m, \tag{5.8}$$

whenever paying attention to that a rooted vertex non-separable tree-like map is a rooted vertex non-separable plane tree iff $m=n$.

The first few terms of $t^N(x)$ are

$$t^N(x) = x + x^2 + 2x^3 + 5x^4 + 14x^5 + \cdots. \tag{5.9}$$

In fact, we always have the relation

$$t^N(x) = xt(x), \tag{5.10}$$

where $t(x)$ is the corresponding enumerating function of rooted plane trees. It is easy to find the 1-to-1 correspondence between rooted vertex non-separable plane trees with m edges and rooted plane trees with $m-1$ edges. In addition, since all the non-separable vertices are with valency 1, the coefficients of x^m $m\geqslant 1$, in (5.8) are also the number of the rooted plane trees of m edges whose rooted vertices are the ones with valency 1.

§ 6. Three Particular Cases

In order to determine the explicit expression of $p^N(x, y)$, according to (3.16), the only thing remaining we have to do is to expand the coefficients of y^k, $k>0$, into power series of x. Let us write

$$p^N(x, y) = \sum_{k=1}^{\infty} p_k^N(x) y^k, \tag{6.1}$$

where

$$p_k^N(x) = \sum_{l=0}^{k-1} B_{l+1}(t) \left[\frac{(2t+1)^2}{(3t+1)}\right]^{k-l-1}. \tag{6.2}$$

In the following, we apply Lagrange's Theorem to find the expansions term by term for $k\leqslant 3$.

$k=1$. In this case, $l=0$, i. e.,

$$p_1^N(x) = B_1(t) = -t(4t+1)(3t+1)^{-1}.$$

Since

$$\frac{d}{dt} B_1(t) = -\frac{12t^2+8t+1}{(3t+1)^2}, \tag{6.3}$$

$$\frac{d^{m-1}}{dt^{m-1}} (-1)^m \frac{t^s}{(3t+1)^{m+2}}\bigg|_{t=0} = \binom{m+1}{s} s!(-1)^{s+1} \frac{(2m-s)!}{(m+1)!} 3^{m-s-1} \tag{6.4}$$

for $s\geqslant 0$, and

$$\frac{d^{m-1}}{dt^{m-1}} (-1)^m (3t+1)^{-m} \frac{d}{dt} B_1(t)\,|_{t=0} = 2\cdot 3^{m-1} m \frac{(2m-2)!}{(m+1)!}, \tag{6.5}$$

we obtain

$$p_1^N(x) = \sum_{m=1}^{\infty} 2\cdot 3^{m-1} \frac{(2m-2)!}{(m-1)!(m+1)!} x^m. \tag{6.6}$$

$k=2$. From (6.2), and (3.17),

$$p_2^N(x) = B_1(t)\frac{(2t+1)^2}{3t+1} = -\frac{16t^4+20t^3+8t^2+t}{(3t+1)^2}.$$

Since

$$\frac{d}{dt} p_2^N(x) = -(96t^4+124t^3+60t^2+13t+1)(3t+1)^{-(m+3)}, \tag{6.7}$$

$$\frac{d^{m-1}}{dt^{m-1}}(-1)^m(3t+1)^{-(m+3)}t^s|_{t=0}=\binom{m-1}{s}s!(-1)^m\frac{d^{m-s-1}}{dt^{m-s-1}}(3t+1)^{-(m+3)}|_{t=0}$$

$$=\begin{cases}0, & m<s+1,\ s\geqslant 0;\\ (-1)^{s+1}s!\binom{m-1}{s}3^{m-s-1}\dfrac{(2m-s+1)!}{(m+2)!}, & m\geqslant s+1,\ s\geqslant 0,\end{cases} \tag{6.8}$$

and after reduction,

$$\frac{d^{m-1}}{dt^{m-1}}(-1)^m(3t+1)^{-m}\frac{d}{dt}p_2^N(x)|_{t=0}$$

$$=\begin{cases}1, & m=1;\\ 4\cdot 3^{m-3}(7m+4)m(m-1)\dfrac{(2m-3)!}{(m+3)!}, & m\geqslant 2,\end{cases} \tag{6.9}$$

we have

$$p_2^N(x)=x+\sum_{m=2}^{\infty}4\cdot 3^{m-3}(7m+4)\frac{(2m-3)!}{(m-2)!(m+1)!}x^m. \tag{6.10}$$

$k=3$. Based on (6.2) and (3.17),

$$p_3^N(x)=\sum_{l=0}^{2}B_{l+1}(t)\left[\frac{(2t+1)^2}{3t+1}\right]^{2-l}$$

$$=-t(4t+1)(3t+1)^{-1}\frac{(2t+1)^4}{(3t+1)^2}+t(2t+1)^2.$$

Let us write

$$\Delta_A(t)=-t(4t+1)\frac{(2t+1)^4}{(3t+1)^3},\ \Delta_B(t)=t(2t+1)^2. \tag{6.11}$$

In what follows, we expand $\Delta_A(t)$, $\Delta_B(t)$ into power series of x by using Lagrange's Theorem respectively.

Because of

$$\frac{d}{dt}\Delta_A(t)=-(576t^6+1248t^5+1104t^4+512t^3+132t^2+18t+1)(3t+1)^{-4} \tag{6.12}$$

and

$$\frac{d^{m-1}}{dt^{m-1}}t^s(3t+1)^{-(m+4)}|_{t=0}=(-1)^{m-s-1}s!\binom{m-1}{s}3^{m-s-1}\frac{(2m-s+2)!}{(m+3)!}, \tag{6.13}$$

for any $s\geqslant 0$, we find

$$\frac{d^{m-1}}{dt^{m-1}}(-1)^m(3t+1)^{-m}\frac{d}{dt}\Delta_A(t)|_{t=0}=576\cdot 3^{m-7}6!\binom{m-1}{6}\frac{(2m-4)!}{(m+3)!}$$

$$-1248\cdot 3^{m-6}5!\binom{m-1}{5}\frac{(2m-3)!}{(m+3)!}+1104\cdot 4!3^{m-5}\binom{m-1}{4}\frac{(2m-2)!}{(m+3)!}$$

$$-512\cdot 3!3^{m-4}\binom{m-1}{3}\frac{(2m-1)!}{(m+3)!}+132\cdot 2!3^{m-3}\binom{m-1}{2}\frac{(2m)!}{(m+3)!}$$

$$-18\cdot 3^{m-2}\binom{m-1}{1}\frac{(2m+1)!}{(m+3)!}+3^{m-1}\frac{(2m+2)!}{(m+3)!}$$

$$=16\cdot 3^{m-4}m(m-1)\frac{(2m-4)!}{(m+3)!}(9(2m-1)(2m-3)(5m^2-24m+28)$$

$$-4(m-2)(m-3)(45m^2-83m+32))$$

$$=16\cdot 3^{m-4}m(m-1)(m^2-4)(8m+3)\frac{(2m-4)!}{(m+3)!},\ m\geqslant 2, \tag{6.14}$$

and it is easily seen that the derivative is 1 when $m=1$.

On the other hand

$$\frac{d^{m-1}}{dt^{m-1}}(-1)^m(3t+1)^{-m}\frac{d}{dt}\Delta_B(t)\big|_{t=0}$$

$$=-12\cdot 3^{m-3}(m-1)(m-2)\frac{(2m-4)!}{(m-1)!}+8\cdot 3^{m-2}(m-1)\frac{(2m-3)!}{(m-1)!}-3^{m-1}\frac{(2m-2)!}{(m-1)!}$$

$$=2\cdot 3^{m-2}\frac{(2m-4)!}{(m-2)!},\ m\geqslant 2, \tag{6.15}$$

and -1, when $m=1$.

In conclusion, according to (6.14) and (6.15), we obtain

$$p_3^N(x)=\sum_{m=1}^{\infty}\frac{d^{m-1}}{dt^{m-1}}(-1)^m(3t+1)^{-m}\frac{d}{dt}(\Delta_A(t)+\Delta_B(t))\big|_{t=0}\frac{x^m}{m!}$$

$$=\sum_{m=2}^{\infty}2\cdot 3^{m-4}(73m^2-68m-21)\frac{(2m-4)!}{(m+3)!(m-2)!}x^m. \tag{6.16}$$

§7. The General Formula of $p^N(x,y)$

From (3.16) and that obtained in the last section, for determining the general formula of $p^N(x, y)$, it is only necessary to expand

$$H_k(t)=\sum_{l=0}^{k-1}B_{l+1}(t)\left[\frac{(2t+1)^2}{2t+1}\right]^{k-l-1} \tag{7.1}$$

into power series of x, for $n\geqslant 4$. In this case, we may write

$$H_k(t)=H_{\text{I}}(k;\ t)+H_{\text{II}}(k;\ t)+H_{\text{III}}(k;\ t), \tag{7.2}$$

where

$$\begin{cases} H_{\text{I}}(k;\ t)=B_1(t)\left[\dfrac{(2t+1)^2}{3t+1}\right]^{k-1}, \\ H_{\text{II}}(k;\ t)=B_3(t)\left[\dfrac{(2t+1)^2}{3t+1}\right]^{k-3}, \\ H_{\text{III}}(k;\ t)=\displaystyle\sum_{l=3}^{k-1}B_{l+1}(t)\left[\dfrac{(2t+1)^2}{3t+1}\right]^{k-l-1}. \end{cases} \tag{7.3}$$

In order to find the expansions of $H_{\text{I}}(k;\ t)$, $H_{\text{II}}(k;\ t)$ and $H_{\text{III}}(k;\ t)$ as power series of x, the following functions

$$\varphi(s,\ r;\ t)=\frac{t^s}{(3t+1)^s},\ s\geqslant 0,\ r>0 \tag{7.4}$$

have to be considered first because all of $H_{\text{I}}(k;\ t)$, $H_{\text{II}}(k;\ t)$ and $H_{\text{III}}(k;\ t)$ may be represented as linear forms of them.

Since

$$\frac{d}{dt}\varphi(s,\ r;\ t)=\frac{t^{s-1}}{(3t+1)^{r+1}}(3(s-r)t+s) \tag{7.5}$$

and

$$\frac{d^{m-1}}{dt^{m-1}}(3t+1)^{-m}\frac{d}{dt}\varphi(s, r; t)\,|_{t=0}$$
$$=(-1)^{m-s}3^{m-s}m(s+r)\frac{(m-1)!}{(m-s)!}\frac{(2m+r-s-1)!}{(m+r)!} \tag{7.6}$$

for $m\geqslant s$; naturally, 0 when $m<s$, we have

$$\varphi(s, r; t)=(-1)^s\sum_{m=s}^{\infty}3^{m-s}\frac{s+r}{m+r}\binom{2m+r-s-1}{m-s}x^m. \tag{7.7}$$

From now on, we find the expansions of $H_{\mathrm{I}}(k; t)$, $H_{\mathrm{II}}(k; t)$ and $H_{\mathrm{III}}(k; t)$ separately by using (7.7).

For $H_{\mathrm{I}}(k; t)$, (3.17) and (7.3) mean that

$$H_{\mathrm{I}}(k; t)=-t(4t+1)(2t+1)^{2(k-1)}(3t+1)^{-k}$$
$$=\left[-2t^2\sum_{j=0}^{2(k-1)}2^j\binom{2k-2}{j}t^j-t\sum_{j=0}^{2k-1}2^j\binom{2k-1}{j}t^j\right](3t+1)^{-k}. \tag{7.8}$$

After rearrangement and reduction,

$$H_{\mathrm{I}}(k; t)=-\varphi(1, k; t)-\sum_{j=2}^{2k}2^{j-1}\frac{2k+j-2}{j-1}\binom{2k-2}{j-2}\varphi(j, k; t). \tag{7.9}$$

By using (7.7), we may obtain

$$H_{\mathrm{I}}(k; t)=x+\sum_{m=2}^{\infty}H^{\mathrm{I}}_{m,k}x^m, \tag{7.10}$$

where

$$H^{\mathrm{I}}_{m,k}=3^{m-1}\frac{k+1}{m+k}\binom{2m+k-2}{m-1}$$
$$+\sum_{j=2}^{\min(m,2k)}(-1)^{j-1}2^{j-1}3^{m-j}\frac{2k+j-2}{j-1}\frac{k+j}{m+j}\binom{2k-2}{j-2}\times\binom{2m+k-j-1}{m-j}, \tag{7.11}$$

for $m\geqslant 1$, $k\geqslant 4$.

To calculate $H_{\mathrm{II}}(k; t)$, from (3.17) and (7.3), we have

$$H_{\mathrm{II}}(k; t)=t(2t+1)^2\left[\frac{(2t+1)^2}{3t+1}\right]^{k-3}$$
$$=\sum_{j=0}^{2k-4}2^j\binom{2k-4}{j}\varphi(j+1, k-3; t). \tag{7.12}$$

On account of (7.7), the following form may be found:

$$H_{\mathrm{II}}(k; t)=\sum_{m=1}^{\infty}H^{\mathrm{II}}_{m,k}x^m, \tag{7.13}$$

where

$$H^{\mathrm{II}}_{m,k}=\sum_{j=1}^{\min(m,2k-3)}(-1)^j2^{j-1}3^{m-j}\frac{k+j-3}{k+m-3}\binom{2k-4}{j-1}\binom{2k+k-j-4}{m-j}, \tag{7.14}$$

for any $m\geqslant 1$, $k\geqslant 4$.

Finally, we detrmine $H_{\mathrm{III}}(k; t)$, which is the most complicated case among the three. According to (3.17) and (7.3),

$$H_{\rm III}(k;\ t)=\sum_{l=3}^{k-1}(A_{l+1}-(t^2+t)A_{l-1}-A_l)\left[\frac{(2t+1)^2}{3t+1}\right]^{k-l-1}. \tag{7.15}$$

For convenience, let us write

$$\begin{cases}\Psi_0(k;\ t)=\sum_{l=3}^{k-1}A_{l+1}\left[\frac{(2t+1)^2}{3t+1}\right]^{k-l-1};\\ \Psi_{(0)}(k;\ t)=\sum_{l=3}^{k-1}A_l\left[\frac{(2t+1)^2}{3t+1}\right]^{k-l-1}\\ \Psi_{\langle s\rangle}(k;\ t)=\sum_{l=3}^{k-1}t^sA_{l-1}\left[\frac{(2t+1)^2}{3t+1}\right],\ s=1,\ 2.\end{cases} \tag{7.16}$$

Then, (7.15) becomes

$$H_{\rm III}(k;\ t)=\Psi_0(k;\ t)-\Psi_{(0)}(k;\ t)-\Psi_{\langle 1\rangle}(k;\ t)-\Psi_{\langle 2\rangle}(k;\ t). \tag{7.17}$$

As above, expressing $\Psi_0(k;\ t)$, $\Psi_{(0)}(k;\ t)$ $\Psi_{\langle 1\rangle}(k;\ t)$ and $\Psi_{\langle 2\rangle}(k;\ t)$ as linear forms of $\varphi(s,\ r;\ t)$ and then employing (7.7), we may derive the following

$$\Psi_0(k;\ t)=\sum_{m=2}^{\infty}\Psi^0_{m,k}x^m, \tag{7.18}$$

where

$$\Psi^0_{m,k}=\sum_{l=3}^{\min(k-1,2m-1)}\Psi_0(m,\ k;\ l), \tag{7.19}$$

$$\Psi_0(m,\ k;\ l)=\sum_{i=\max(1,l+1-m)}^{\lfloor (l+1)/2\rfloor}F(l+1,\ i)G_0(m,\ k;\ l,\ i), \tag{7.20}$$

$$G_0(m,\ k;\ l,\ i)=\sum_{j=l-i+1}^{\min(m,2k-l-1)}(-1)^j2^{j+i-l-1}3^{m-j}\times\frac{j+k-l-1}{m+k-l-1}\binom{2k+i-2l-2}{j+i-l-1}\binom{2m+k-j-l-2}{m-j}. \tag{7.21}$$

$$\Psi_{(0)}(k;\ t)=\sum_{m=2}^{\infty}\Psi^{(0)}_{m,k}x^m, \tag{7.22}$$

where

$$\Psi^{(0)}_{m,k}=\sum_{l=3}^{\min(k-1,2m)}\Psi_{(0)}(m,\ k;\ l), \tag{7.23}$$

$$\Psi_{(0)}(m,\ k;\ l)=\sum_{i=\max(1,l-m)}^{\lfloor l/2\rfloor}F(l,\ i)G_{(0)}(m,\ k;\ l,\ i), \tag{7.24}$$

$$G_{(0)}(m,\ k;\ l,\ i)=\sum_{j=l-i}^{\min(m,2k-l-2)}(-1)^j2^{j+i-l}3^{m-j}\times\frac{j+k-l-1}{m+k-l-1}\binom{2k+i-2l-2}{j+i-l}\binom{2m+k-j-l-2}{m-j}. \tag{7.25}$$

And, for $s=1,\ 2$,

$$\Psi_{\langle s\rangle}(k;\ t)=\sum_{m=s+1}^{\infty}\Psi^{\langle s\rangle}_{m,k}x^m, \tag{7.26}$$

where

$$\Psi^{\langle s\rangle}_{m,k}=\sum_{l=3}^{\min(k-1,2(m-s)+1)}\Psi_{\langle s\rangle}(m,\ k;\ l), \tag{7.27}$$

$$\Psi_{\langle s\rangle}(m,\ k;\ l)=\sum_{i=\max(1,l+s-m-1)}^{\lfloor l/2\rfloor}F(l-1,\ i)G_{\langle s\rangle}(m,\ k;\ l,\ i), \tag{7.28}$$

$$G_{\langle s\rangle}(m, k; l, i)=\sum_{j=l+s-i-1}^{\min(m,2k+s-l-3)}(-1)^{j}2^{j+i-l-s+1}3^{m-j}$$
$$\times\frac{j+k-l-1}{m+k-l-1}\binom{2k+i-2l-2}{j+i-l-s+1}\binom{2m+k-l-j-2}{m-j}. \quad (7.29)$$

In all of (7.20), (7.24) and (7.28), there is

$$F(l, i)=(-1)^{i+l}\frac{1}{i}\binom{2i-2}{i-1}\binom{l-1}{l-2i}. \quad (7.30)$$

In brief, from those obtained above, we have

$$H_{\mathrm{III}}(k; t)=\sum_{m=2}^{\infty}H_{m,k}^{\mathrm{III}}x^m, \quad (7.31)$$

where

$$H_{m,k}^{\mathrm{III}}=\Psi_{m,k}^{0}-\Psi_{m,k}^{(0)}-\Psi_{m,k}^{\langle 1\rangle}-\Psi_{m,k}^{\langle 2\rangle}. \quad (7.32)$$

Noticing that no planar maps have $m<\lceil k/2\rceil$, of course, which may be examined directly or indirectly, we are allowed only to consider $m\geqslant\lceil k/2\rceil$. In this case, we have

$$H(k; t)=\sum_{m=\lceil k/2\rceil}^{\infty}H_{m,k}^{N}x^m,$$

where

$$H_{m,k}^{N}=H_{m,k}^{\mathrm{I}}+H_{m,k}^{\mathrm{II}}+H_{m,k}^{\mathrm{III}}, \quad m\geqslant\lceil k/2\rceil,$$

in which the three terms are determined by (7.11), (7.14) and (7.32) respectively.

§8. An Example

Using the general formula of $p^N(x, y)$ obtained in § 7, we calculate $H(4; t)$ as an example.

In order to find the first 7 terms of $H_{\mathrm{III}}(4, t)$, we obtain the following table by (7.19—30).

m	2	3	4	5	6	7
$\Psi_{m,4}^{0}$	1	5	34	273	2394	22194
$\Psi_{m,4}^{(0)}$	2	8	54	432	3780	34992
$\Psi_{m,4}^{\langle 1\rangle}$	−1	−4	−27	−216	−1890	−17496
$\Psi_{m,4}^{\langle 2\rangle}$	0	1	7	57	504	4698

Thus, it can be seen that

$$H_{m,4}^{\mathrm{III}}=\Psi_{m,4}^{0}-\Psi_{m,4}^{(0)}-\Psi_{m,4}^{\langle 1\rangle}-\Psi_{m,4}^{\langle 2\rangle}=0, \quad m\leqslant 7. \quad (8.1)$$

In fact, from (7.3) and (3.17), we have

$$H_{\mathrm{III}}(4; t)=A_4-(t^2+t)A_2-A_3$$
$$=-3t^3(2t+1)+t^2(2t+1)^2+(t^2+t)(2t+1)t-2(2t+1)t^2=0. \quad (8.2)$$

Hence, in this case,

$$H(4;\ t)=H_{\mathrm{I}}(4;\ t)+H_{\mathrm{II}}(4;\ t). \tag{8.3}$$

According to (7.11) and (7.14), after rearrangement and reduction, we finally find

$$H(4;\ t)=x^2+\sum_{m=3}^{\infty}8\cdot 3^{m-7}(856m^4+3456m^3+695m^2-5040m-936)\times\frac{(2m-5)!}{(m+4)!\,(m-3)!}\,x^n. \tag{8.4}$$

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