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GRADED MODULES OF GRADED LIE ALGEBRAS OF CARTAN TYPE (III) ——IRREDUCIBLE MODULES**

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Abstract

Over a field of characteristic $\neq 2, 3$, all irreducible positive and negative graded modules of simple Lie algebras $L(n)$ and $L(n, m)$ of Cartan type W, S and H are determined. Further, all irreducible positive and negative filtered modules of $L(n, m)$ are determined. For $L(n)$, every irreducible negative filtered module is a negative graded module, but there exist irreducible positive filtered modules which are not graded.

§ 0. Introduction

Let F be an algebraically closed field, Char $F\neq 2, 3$, L a (finite-dimensional or infinite-dimensional) simple graded Lie algebra of Cartan type W, S or H. We first determine all irreducible positive and negative graded modules of L. Let $\widetilde{M}(V_0)$ be the unique irreducible positive graded L-module with the irreducible L_0-module V_0 as its base space. Then $\widetilde{M}(V_0)=\widetilde{V}_0$ unless $V_0=V_0(\lambda_i)$, the highest weight module with a fundamental weight λ_i as its highest weight, for some i. When $V_0=V_0(\lambda_i)$, write $\widetilde{V}_0=\widetilde{V}_0(\lambda_i)$. Then $\widetilde{V}_0(\lambda_i)$ is identied with the L-module consisting of all differential forms of i-th degree. Using exterior differential operation we can determine $\widetilde{M}(V_0(\lambda_i))$ which is the minimal submodule of $\widetilde{V}_0(\lambda_i)$. When L is finite dimensional, we calculate the dimensions of $\widetilde{M}(V_0(\lambda_i))$. By duality, all irreducible negative graded modules of L are also determined.

We further define and discuss the positive and negative filtered modules of L. For finite-dimensional $L=L(n, m)$, all irreducible positive and negative filtered modules are determined. For infinite-dimensional $L=L(n)$, the irreducible negative filtered modules are also determined: they are just the irreducible negative graded modules. On the contrary, there exist irreducible positive filtered modules which are not graded. To determine all irreducible positive filtered modules of $L(n)$ remains an open question.

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§1. Irreducibility of $\widetilde{V}_0$ and $\underset{\sim}{V}_0$

We adopt all the notations used in [4] and [5]. If ρ_0 is a representation of L_0 in V_0, let $\tilde{\rho}_0$ and $\underset{\sim}{\rho}_0$ denote the representations of L in $\widetilde{V}_0$ and $\underset{\sim}{V}_0$ respectively. Let Z denote the one-dimensional trivial module, $T|_V$ (or simply T) denote the trace representation of $gl(n)$ (and its subalgebras) in the space V: $T(A)=(\mathrm{Tr}A)\cdot\mathbf{1}_V$, $A\in gl(n)$. As in [4], if $D=\sum_i a_iD_i\in L$, let $\widetilde{D}=\sum_{i,j}D_i(a_j)\otimes E_{ij}\in\mathfrak{A}\otimes L_0$. Let

$$\widetilde{M}(V_0)=(\widetilde{V}_0)_{\min},\tag{1.1}$$

if V_0 is L_0-irreducible. (To distinguish finite-dimensional and infinite-dimensional cases if necessary, $\widetilde{M}(V_0)$ will be denoted by $\widetilde{M}(V_0,\,n,\,\boldsymbol{m})$ and $\widetilde{M}(V_0,\,n)$ respectively). $\widetilde{M}(V_0)$ is the unique irreducible positive graded module with V_0 as its base space.

We first assume $L=L(n,\,\boldsymbol{m})$.

Lemma 1.1 *Let ρ_0 be a representation of L_0 in the irreducible module V_0. If V_0 is not a highest weight module, then there exists a root vector X of L_0 such that $\rho_0(X)$ and $\tilde{\rho}_0(X)$ have unique characteristic value $a\neq0$ in V_0 and $\widetilde{V}_0$ respectively.*

Proof V_0 is not graded. By [5, Lemma 1.4], we may assume $\rho_0(\mathfrak{n}^+)$ is not nilpotent, where $\mathfrak{n}^+$ is the subspace of L_0 spanned by the positive root vectors. Let P be the set of positive root vectors of L_0. Then $\rho_0(P)$ is a weakly closed set. If all $\rho_0(Y)$, $Y\in P$, are nilpotent, then $\rho_0(\mathfrak{n}^+)$ is nilpotent[2], a contradiction. Hence we have $X\in P$ such that $\rho_0(X)$ is not nilpotent. Now, $\rho_0(X)^p-\rho_0(X^{[p]})$ commutes with $\rho_0(L_0)$ and $X^{[p]}=0$ in L_0. We have $\rho_0(X)=a'\cdot\mathbf{1}_{V_0}$, $a'\neq0$, and $a=a'^{1/p}$ is the unique characteristic value of $\rho_0(X)$. Similarly, $\tilde{\rho}_0(X)^p=\delta(X)\otimes\mathbf{1}_{V_0}+\mathbf{1}_{\mathfrak{A}}\otimes\rho_0(X)$ and $\delta(X)^p=0$ since δ is a restricted representation. Hence $\rho_0(X)^p=\mathbf{1}_{\mathfrak{A}}\otimes a'\cdot\mathbf{1}_{V_0}=a'\cdot\mathbf{1}_{\widetilde{V}_0}$.

Proposition 1.1. *Let V_0 be an irreducible L_0-module. If V_0 is not a highest weight module, then $\widetilde{V}_0(n,\,\boldsymbol{m})$ is irreducible.*

Proof Let X be as in Lemma 1.1. We have $l(\widetilde{M}(V_0))\geqslant N(\boldsymbol{m})-2$ (for $L=H(n,\,\boldsymbol{m})$, cf. [5, Lemma 2.6]). Hence there is a non-zero $v=\sum_{|\alpha|=N(\boldsymbol{m})-2}x^{(\alpha)}\otimes v_\alpha\in\widetilde{M}(V_0)$ such that $\tilde{\rho}_0(X)v=av$. Denote

$$x_i=x^{(\varepsilon_i)}.\tag{1.2}$$

(I) $L=S(n,\,\boldsymbol{m})$. Then $X=x_iD_j$ (identified with E_{ij}), $i\neq j$. Let $s=\min\{\alpha_i\,|\,v_\alpha\neq0\}$.

1) $s=\pi_i$, that is, we have $v_\alpha\neq0$, $\alpha=\pi-\varepsilon_k-\varepsilon_l$, k, $l\neq i$, for some α. If $k\neq j$, $l\neq j$, let $Y=x_ix_kx_lD_j$; if $k=j$, $l\neq j$, let $Y=x_ix_lx_jD_j-x_i^{(2)}x_lD_i$; if $k=l=j$, let $Y=x_ix_j^{(2)}D_j-x_jx_i^{(2)}D_i$. Then $Y\in S(n,\,\boldsymbol{m})$. Direct computation shows $\tilde{\rho}_0(Y)v=ax^{(\pi)}\otimes v_\alpha\neq0$.

2) $s=\pi_i-1$, i. e., there is $v_\alpha\neq0$, $\alpha=\pi-\varepsilon_i-\varepsilon_k$, $k\neq i$. If $k=j$, let $Y=x_ix_j^{(2)}D_j$,

$-x_i^{(2)}x_jD_i$, then $\tilde{\rho}_0(Y)v=ax^{(\pi)}\otimes v_\alpha\neq 0$. If $k\neq j$, let $Y=2x_i^{(2)}x_k D_j$, theh $\tilde{\rho}_0(Y)v=ax^{(\pi)}\otimes v_\alpha+x^{(\pi)}\otimes\rho_0(E_{ij})v_\alpha$ which is not equal to zero since otherwise we would have $\rho_0(E_{ij})v_\alpha=-av_\alpha$ contrary to Lemma 1.1.

3) $s=\pi_i-2$, i. e., there is $v_\alpha\neq 0$, $\alpha=\pi-2\varepsilon_i$. Let $Y=3x_i^{(3)}D_j$. Then $\tilde{\rho}_0(Y)v=ax^{(\pi)}\otimes v_\alpha+2x^{(\pi)}\otimes\rho_0(E_{ij})v_\alpha\neq 0$ since $-a/2$ is not a characteristic value of $\rho_0(E_{ij})$.

Now in all cases we have $\tilde{M}(V_0)\cap(\tilde{V}_0)_{N(m)}\neq 0$. Since $(\tilde{V}_0)_{N(m)}$ is L_0-irreducible ([5, Lemma 2.4]), $\tilde{M}(V_0)\supset(\tilde{V}_0)_{N(m)}$. Hence $\tilde{M}(V_0)=\tilde{V}_0$ by [5, Lemma 2.1].

(II) $L=W(n, \boldsymbol{m})$. The proof is similar to (I).

(III) $L=H(n, \boldsymbol{m})$, $n=2r$. Let

$$\sigma(i)=\begin{cases}1, & 1\leqslant i\leqslant r,\\ -1, & r<i\leqslant n,\end{cases}\tag{1.3}$$

$$\tilde{i}=i+\sigma(i)r,\ i=1, \cdots, n.\tag{1.4}$$

Then $H(n, \boldsymbol{m})$ is spanned by

$$B(f):=\sum_{i=1}^{n}\sigma(\tilde{i})(D_{\tilde{i}}f)D_i,\tag{1.5}$$

where $f\in\mathfrak{A}(n, \boldsymbol{m})$, $\deg f<|\pi|$. The root vectors of $L_0\cong sp(n)$ are multiples of $B(x_ix_j)$, $i\neq\tilde{j}$, j, i, $j=1, \cdots, 2v$, and $B(x_i^{(2)})$, $i=1, \cdots, 2r$ (identified with $\sigma(i)E_{j\tilde{i}}+\sigma(j)E_{i\tilde{j}}$ and $\sigma(i)E_{i\tilde{i}}$ respectively, cf. [6, p. 12]).

(1) $X=B(x_ix_j)=\sigma(i)x_jD_{\tilde{i}}+\sigma(j)D_{\tilde{i}}$, $i\neq j$, $\tilde{j}$. Let $s=\min\{\alpha_i, \alpha_j|v_\alpha\neq 0\}$. Since the situations of i and j are symmetrical, we may assume $s=\alpha_i$ for some α.

1) $s=\pi_i$, i. e., $v_\alpha\neq 0$, $\alpha=\pi-\varepsilon_k-\varepsilon_l, k$, $l\neq i$, j. Let $Y=B(x_kx_lx_ix_j)$. Direct computation shows $\tilde{\rho}_0(Y)v=ax^{(\pi)}\otimes v_\alpha$

2) $s=\pi_i-1$, i. e., $v_\alpha\neq 0$, $\alpha=\pi-\varepsilon_i-\varepsilon_k$, $k\neq i$. If $k=j$, let $Y=2B(x_i^{(2)}x_j^{(2)})$; if $k\neq j$, let $Y=2B(x_i^{(2)}x_jx_k)$. Then $\tilde{\rho}_0(Y)v=ax^{(\pi)}\otimes v_\alpha+x^{(\pi)}\otimes\rho_0(X)v_\alpha\neq 0$.

3) $s=\pi_i-2$, i. e., $v_\alpha\neq 0$, $\alpha=\pi-2\varepsilon_i$. Let $Y=3B(x_i^{(3)}x_j)$. Then $\tilde{\rho}_0(Y)v=ax^{(\pi)}\otimes v_\alpha+2x^{(\pi)}\otimes\rho_0(X)v_\alpha\neq 0$.

(2) $X=B(x_i^{(2)})=\sigma(i)x_iD_{\tilde{i}}$. Let $s=\min\{\alpha_i|v_\alpha\neq 0\}$.

1) $s=\pi_i$, i. e., there is $v_\alpha\neq 0$, $\alpha=\pi-\varepsilon_k-\varepsilon_l$, k, $l\neq i$. Let $Y=B(x_kx_lx_i^{(2)})$. Then $\tilde{\rho}_0(Y)v=ax^{(\pi)}\otimes v_\alpha\neq 0$.

2) $s=\pi_i-1$, i. e., there is $v_\alpha\neq 0$, $\alpha=\pi-\varepsilon_i-\varepsilon_k$, $k\neq i$. Let $Y=2B(x_i^{(3)}x_k)$. Then $\tilde{\rho}(Y)v=ax^{(\pi)}\otimes v_\alpha+x^{(\pi)}\otimes\rho_0(X)v_\alpha\neq 0$.

3) $s=\pi_i-2$, i. e., there is $v_\alpha\neq 0$, $\alpha=\pi-2\varepsilon_i$. Let $Y=3B(x_i^{(4)})$. Then $\tilde{\rho}_0(Y)=ax^{(\pi)}\otimes v_\alpha+2x^{(\pi)}\otimes\rho_0(X)v_\alpha\neq 0$.

In all cases we have $\tilde{M}(V_0)\cap(\tilde{V})_{N(m)}\neq 0$ and $\tilde{M}(V_0)=\tilde{V}_0$.

Let $\mathfrak{h}(L_0)$ be the standard Cartan subalgebra of $L_0=gl(n)$, $sl(n)$ or $sp(n)$ and Λ_i, $i=1, \cdots, n$, the linear functions on $\mathfrak{h}(gl(n))=\langle E_{11}, \cdots, E_{nn}\rangle$ such that

$$\Lambda_i(E_{jj})=\delta_{ij}.\tag{1.6}$$

The restriction of Λ_i on every $\mathfrak{h}(L_0)$ will also be denoted by Λ_i. Let

$$\lambda_0=0,\ \lambda_i=\sum_{j=1}^{i}\Lambda_j,\ i=1,\ \cdots,\ n. \tag{1.7}$$

Then $\lambda_i(i=1, \cdots, n$ for $L_0=gl(n)$; $i=1, \cdots, n-1$ for $L_0=sl(n)$; $i=1, \cdots, r$ for $L_0=sp(n))$ are the fundamental weights of L_0. Every weight of $sl(n)$ or $sp(n)$ is a linear combination of fundamental weights[6]. We have

$$\Lambda_i=\lambda_i-\lambda_{i-1},\ i=1,\ \cdots,\ n. \tag{1.7$'$}$$

Proposition 1.2. *Let V_0 be a highest weight module of L_0 with highest weight $\lambda\neq 0$. If λ is not a fundamental weight, then $\tilde{V}_0(n, \boldsymbol{m})$ is irreducible.*

Proof Let v_λ be a highest weight vector of V_0.

(I) $L=S(n, \boldsymbol{m})$. Let

$$h_i=E_{ii}-E_{i+1,i+1},\ i=1,\ \cdots,\ n-1. \tag{1.8}$$

We have

$$\lambda_i(h_j)=\delta_{ij},\ i,\ j=1,\ \cdots,\ n-1. \tag{1.9}$$

The positive roots of $sl(n)$ are $\lambda_i-\lambda_j$, $i<j$; the corresponding positive root vectors are E_{ij}, $i<j$. Let

$$A_{ij}(f)=(D_if)D_j-(D_jf)D_i,\ f\in\mathfrak{A}(n,\ \boldsymbol{m}),\ i,\ j=1,\ \cdots,\ n. \tag{1.10}$$

Then $A_{ij}(f)\in L$[3].

Let the highest weight $\lambda=\sum_{k=1}^{n-1}c_k\lambda_k$, $c_k\in F$.

(1) If $c_i\neq 0$, 1 for some i, let $A=A_{i+1,i}(x^{(\pi)})$. Then

$$u:=\tilde{\rho}_0(A)(1\otimes v_\lambda)=c_ix^{(\pi-\varepsilon_i-\varepsilon_{i+1})}\otimes v_\lambda+\sum_{k>i}x^{(\pi-\varepsilon_{i+1}-\varepsilon_k)}\otimes\rho_0(E_{k,i})v_\lambda$$
$$-\sum_{k>i+1}x^{(\pi-\varepsilon_i-\varepsilon_k)}\otimes\rho_0(E_{k,i+1})v_\lambda\neq 0.$$

Let $B=A_{i+1,i}(x_i^{(2)}x_{i+1}^{(2)})$. Then $\tilde{\rho}_0(B)u=x^{(\pi)}\otimes(c_i^2-c_i)v_\lambda\neq 0$.

(2) Suppose all $\lambda_k=0$ or 1. If there are more than one $c_k=1$, let the foremost two be c_i and c_j, $i<j$. Let $A=A_{i,j+1}(x^{(\pi)})$, $B=A_{i,j+1}(x_i^{(2)}x_{j+1}^{(2)})$ and $h=h_i+\cdots+h_j$. Direct computation shows $\tilde{\rho}_0(B)\tilde{\rho}_0(A)(1\otimes v_\lambda)=(\lambda(h)^2-\lambda(h))x^{(\pi)}\otimes v_\lambda\neq 0$ since $\lambda(h)=2$.

Thus $\widetilde{M}(T_0)\cap(\tilde{V}_0)_{N(\boldsymbol{m})}\neq 0$ and $\widetilde{M}(V_0)=\tilde{V}_0$.

(II) $L=W(n, \boldsymbol{m})$. Suppose $\lambda=\sum_{k=1}^{n}c_k\lambda_k$. If $\tilde{V}_0$ is reducible, by the discussion of (I), we may assume the restriction of λ on $\mathfrak{h}(sl(n))$ is a fundamental weight λ_i, $1\leqslant i\leqslant n-1$, that is $\lambda=a\sum_{j=1}^{i}\Lambda_j+(a-1)\sum_{j=i+1}^{n}\Lambda_j, a\in F$. We proceed to show $a=1$. Suppose $a\neq 1$. When $a\neq 0$, let $A=x^{(\pi)}D_1$, $B=X_1^{(2)}D_1$ and $\tilde{\rho}_0(B)\tilde{\rho}_0(A)(\mathbf{1}\otimes v_\lambda)=(a-a^2)v^{(\pi)}\otimes v_\lambda\neq 0$; when $a=0$, let $A=x^{(\pi)}D_n$, $B=x_n^{(2)}D_n$ and $\tilde{\rho}_0(B)\tilde{\rho}_0(A)(\mathbf{1}\otimes v_\lambda)=((a-1)-(a-1)^2)x^{(\pi)}\otimes v_\lambda\neq 0$. It follows that $\widetilde{M}(V_0)=\tilde{V}_0$, i. e., $\tilde{V}_0$ is irreducible, a contradiction.

(III) $L=H(n, \boldsymbol{m})$. Let $\lambda=\sum_{k=1}^{r}c_k\lambda_k=\sum_{k=1}^{r}s_k\Lambda_k$, where $s_k=\sum_{j=1}^{k}c_j$.

The positive roots of $sp(2r)$ are $\Delta_i-\Delta_j$, $1\leqslant i<j\leqslant r$, $\Delta_i+\Delta_j$, $1\leqslant i$, $j\leqslant r$, and $2\Delta_i$, $1\leqslant i\leqslant r$. If μ is a root, let h_μ denote the corresponding coroot and e_μ the corresponding standad root vector. For example, $h_{2\Delta_i}=E_{ii}-E_{\tilde{i}\tilde{i}}$, $e_{2\Delta_i}=E_{i\tilde{i}}$, etc. (for details cf. [6]). Set $A=B(x^{(\pi)})$. Then $\tilde{\rho}_0(A)(1\otimes v_\lambda)=\sum_{j=1}^{r}x^{(\pi-\varepsilon_j-\varepsilon_{\tilde{j}})}\otimes\lambda(h_{2\pi_j})v_\lambda+\sum_{\substack{i,j=1\\ i\neq j}}^{r}x^{(\pi-\varepsilon_i-\varepsilon_{\tilde{j}})}\otimes\rho_0(e_{-(\Delta_i+\Delta_j)})v_\lambda+\sum_{1\leqslant j<i\leqslant r}x^{(\pi-\varepsilon_i-\varepsilon_j)}\otimes\rho_0(e_{\Delta_i-\Delta_j})v_\lambda$.

1) If $s_i\neq 0$, 1 for some i, let $B=B(x_i^{(2)}x_{\tilde{i}}^{(2)})$. We have $\tilde{B}=x^{(\varepsilon_i+\varepsilon_{\tilde{i}})}\otimes h_{2\Delta_i}+x^{(2\varepsilon_i)}\otimes e_{-2\Delta_i}-x^{(2\varepsilon_{\tilde{i}})}\otimes e_{2\Delta_i}$ and $\tilde{\rho}_0(B)\tilde{\rho}_0(A)(1\otimes V_\lambda)=(\lambda(h_{2\Delta_i})^2-\lambda(h_{2\Delta_i}))x^{(\pi)}\otimes v_\lambda\neq 0$ since $\lambda(h_{2\Delta_i})=s_i$.

2) Suppose all $s_k=0$ or 1, but $s_i=0$, $s_j=1$, for some $1\leqslant i<j\leqslant r$. Let $B=B(x_ix_{\tilde{i}}x_jx_{\tilde{j}})$. We have $\tilde{\rho}_0(B)\tilde{\rho}_0(A)(1\otimes V_\lambda)=x^{(\pi)}\otimes(\lambda(h_{\Delta_i-\Delta_j})-\lambda(h_{\Delta_i+\Delta_j}))v_\lambda\neq 0$ since $\lambda(h_{\Delta_i-\Delta_j})-\lambda(h_{\Delta_i+\Delta_j})=-2\neq 0$.

Thus, unless $\lambda=\sum_{k=1}^{i}\Delta_k=\lambda_i$, we have $\tilde{M}(V_0)=\tilde{V}_0$.

We now consider the infinite-dimensional cases.

Proposition 1.3. *Let* Char $F=p>3$, $L(n)=W(n)$, $S(n)$ or $H(n)$ *and* V_0 *be an irreducible* L_0*-module. If* V_0 *is not a highest weight module with a fundamental weight* λ_i *as its highest weight, then* $\tilde{V}_0(n)$ *is irreducible.*

Proof For any $\boldsymbol{m}$,

$$(\tilde{V}_0(n))_2=(\tilde{V}_0(n,\boldsymbol{m}))_2\subset\mathscr{U}(L(n,\boldsymbol{m}))(1\otimes V_0)\subset\mathscr{U}(L(n))(1\otimes V_0),$$

and $\tilde{V}_0(n)$ is irreducible by [5, Theorem 2.5].

Proposition 1.4. *Suppose* Char $F=0$, $L(n)$, V_0 *are as in Proposition* 1.3 *and* dim $V_0<\infty$. *Then* $\tilde{V}_0(n)$ *is irreduciblh.*

Proof If $L=S(n)$ or $H(n)$, then V_0 is an integral module ([5, Note 2.2]). Choose a sufficiently large prime p such that V_{0K} is L_0-irreducible, where K is an algebraically closed field of characteristic p. If $\tilde{V}_0(n)$ is reducible, then by a mod p process we should conclude that $\tilde{V}_0(n)_K$ is $L(n)_K$-reducible, which is contrary to Proposition 1.3. When $L=W(n)$, as in Proposition 1.2 (II), we may assume V_0 is a highest weight module with highest weight $\lambda=a\sum_{j=1}^{i}\Delta_j+(a-1)\sum_{j=i+1}^{n}\Delta_j$. As an $S(n)$-module, $\tilde{V}_0$ has a unique proper submodule M (see Theorem 2.4 of the next section). If $a\neq 1$, an easy computation shows $\mathscr{U}(W(n))M\otimes M$.

Let $V_0(\lambda_i)$ denote the highest weight module with highest weight λ_i, ρ_{0i} the corresponding representation and $V_0(\lambda_i)^*$ (ρ_{0i}^*) the dual module (representation) of $V_0(\lambda_i)$ (ρ_{0i}). We have

$$\rho_{0i}^*\cong\rho_{0,n-i}-T. \tag{1.11}$$

Especially if $L_0=sl(n)$ or $sp(n)$, $\rho_{0i}^*\cong\rho_{0,n-i}$. Moreover, when $L_0=sp(n)$, $\rho_{0i}^*\cong\rho_{0i}$. By duality (cf. [5, Corollary 3.4]), we have the following proposition.

Proposition 1.5. *Let Char $F\neq 2, 3$, $L=L(n, \boldsymbol{m})$ or $L(n)$ is of Cartan type W, S or H and ρ_0 an irreducible finite-dimensional representation of L_0. Then $\tilde{\rho}_0$ is irreducible unless $\rho_0=\rho_{0i}^*$, i. e., $\rho_0=\rho_{0i}$ if L is of type S or H, $\rho_0=\rho_{0i}-T$ if L is of type W, for some i.*

§2. Decomposition of $\tilde{V}_0(\lambda_i)$

If $V_0=V_0(\lambda_i)$, $\tilde{V}_0(n, \boldsymbol{m})$ and $\tilde{V}_0(n)$ will be denoted by $\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$ and $\tilde{V}_0(\lambda_i, n)$ respectively (sometimes simply $\tilde{V}_0(\lambda_i)$ for both). Similarly, $\tilde{M}(V_0, n, \boldsymbol{m})$ and $\tilde{M}(V_0, n)$ will be denoted by $\tilde{M}(\lambda_i, n, \boldsymbol{m})$ and $\tilde{M}(\lambda_i, n)$ respectively (or simply $\tilde{M}(\lambda_i)$ for both). It is well-known that $V_0(\lambda_i)$ is isomorphic to the module consisting of all i-vectors, i. e., $V_0(\lambda_i)\cong\wedge^i V_0(\lambda_1)$ where $V_0(\lambda_1)$ is the module of the natural represenation of L_0 and $\wedge^i$ denotes the i-th exterior product[6]. Let $\{e_1, \cdots, e_n\}$ be the natural basis of $V_0(\lambda_1)$, i. e., it satisfies $E_{ij}e_k=\delta_{jk}e_i$. Let $\Omega^i(n)$ and $\Omega^i(n, \boldsymbol{m})$ be the $L(n)$ and $L(n, \boldsymbol{m})$-modules of differential forms of i-th degree with coefficients in $\mathfrak{A}(n)$ and $\mathfrak{A}(n, \boldsymbol{m})$ respectively[3,7]. The linear map $x^{(\alpha)}\otimes(e_{j_1}\wedge\cdots\wedge e_{j_i})\mapsto x^{(\alpha)}dx_{j_1}\wedge\cdots\wedge dx_{j_i}$ is a module isomorphism of $\tilde{V}_0(\lambda_i, n)$ or $\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$ onto $\Omega^i(n)$ or $\Omega^i(n, \boldsymbol{m})$. In the following we shall identify $\tilde{V}_0(\lambda_i, n)$ ($\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$) with $\Omega^i(n)$ ($\Omega^i(n, \boldsymbol{m})$).

Let $I=\{1, \cdots, n\}$, $S\subset I$, and the elements of S be $i_1<i_2<\cdots<i_k$. Write

$$\omega_S=dx_{i_1}\wedge\cdots\wedge dx_{i_k}, \tag{2.1}$$

$$x^{(\pi_S)}=x_{i_1}^{(\pi_{i_1})}\cdots x_{i_k}^{(\pi_{i_k})}. \tag{2.2}$$

Define as usual the exterior differential operation d_i: $\tilde{V}_0(\lambda_i, n)\to\tilde{V}_0(\lambda_{i+1}, n)$ by

$$d_i(f\omega_S)=(df)\wedge\omega_S,\ f\in\mathfrak{A}(n),\ \text{Card}\,S=i, \tag{2.3}$$

where

$$df=\sum_{i=1}^{n}(D_if)\,dx_i. \tag{2.3$'$}$$

It is well-known that

$$\ker d_i=d_{i-1}\tilde{V}_0(\lambda_{i-1}, n). \tag{2.4}$$

Let $d_i(\boldsymbol{m})$ denote the restriction of d_i on $\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$.

Proposition 2.1. *Let $L=W(n, \boldsymbol{m})$. Then (1) $d_{i-1}\tilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})$ is a proper submodule, with length $N(\boldsymbol{m})-1$, of $\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$; (2) $\tilde{M}(\lambda_i, n, \boldsymbol{m})=d_{i-1}\tilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})$, $i=1, \cdots, n$; (3) $\ker d_i(\boldsymbol{m})/d_{i-1}\tilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})$ is a trivial module.*

Proof (1) Write $U=d_{i-1}\tilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})$. Since $d_{i-1}(\tilde{V}(\lambda_{i-1}))_0=0$, we have $l(U)<N(\boldsymbol{m})$. On the other hand, $U_{N(\boldsymbol{m})-1}=\langle d_{i-1}(x^{(\pi)}\omega_S)\mid \text{Card}\,S=i-1\rangle\neq 0$. Hence $l(U)=N(\boldsymbol{m})-1$. (2) $U\cong\tilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})/\ker d_{i-1}(\boldsymbol{m})$ whose top space is isomorphic to $x^{(\pi)}\otimes V_0(\lambda_{i-1})$ and hence is L_0-irreducible. The length of $\tilde{M}(\lambda_i)$ is at least $N(\boldsymbol{m})-1$ ([5, Lemma 2.2]) and so $\tilde{M}(\lambda_i)\supset U_{N(\boldsymbol{m})-1}=\langle d_{i-1}(x^{(\pi)}\omega_S)\rangle$. Applying $\tilde{\rho}_0(D_i)$, $i=1$,

$\cdots$, n, repeatedly, we obtain all $d_{i-1}(x^{(\alpha)}\omega_S)\in\widetilde{M}(\lambda_i)$, i. e., $\widetilde{M}(\lambda_i)=U$. (3) We have $\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})\supsetneqq\ker d_i(\boldsymbol{m})\supset U$ and $l(\ker d_i(\boldsymbol{m}))=N(\boldsymbol{m})-1=l(U)$. It is easily seen that $U_0=d_{i-1}\widetilde{V}(\lambda_{i-1})_1=\widetilde{V}_0(\lambda_i)_0=(\ker d_i(\boldsymbol{m}))_0$. Hence $l(\ker d_i(\boldsymbol{m})/U)<N(\boldsymbol{m})-1$, and $\ker d_i(\boldsymbol{m})/U$ is trivial by [5, Lemma 2.2].

Lemma 2.1. $\ker d_i(\boldsymbol{m})=d_{i-1}\widetilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})\otimes\langle x^{(\pi_S)}\omega_S\,|\,\mathrm{Card}\,S=i\rangle$.

Proof "$\supset$" is obvious. Now we prove "$\subset$". Let $u\in\ker d_i(\boldsymbol{m})$. By (2.4), we may assume (i) $u=d_{i-1}x^{(\alpha)}\omega_T$, Card $T=i-1$. Moreover, we may assume (ii) $x^{(\alpha)}\overline{\in}\mathfrak{A}(n, \boldsymbol{m})$ and u cannot be expressed as $d_{i-1}w$, $w\in\widetilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})$. By Proposition 2.1 (3), $\widetilde{\rho}_0(D_j)u=d_{i-1}(x^{(\alpha-\delta_j)}\omega_T)\in d_{i-1}\widetilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})$, i. e., $x^{(\alpha-\varepsilon_j)}\in\mathfrak{A}(n, \boldsymbol{m})$. Hence we may also assume (iii) $\alpha_k\leqslant\pi_k$, $k\neq j$, $\alpha_j=\pi_k+1$, for some j. A closer observation shows $u=x^{(\pi_S)}\omega_S(=\pm d_{i-1}(x^{(\pi_S+\varepsilon_j)}\omega_{S\setminus\{j\}}), j\in S)$.

Let

$$\dim d_i\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})=a_i. \tag{2.5}$$

By Lemma 2.1, $\dim\ker d_i(\boldsymbol{m})=a_{i-1}+C_i^n$. We have the exact sequence

$$0\to\ker d_i(\boldsymbol{m})\to\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})\to d_i\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})\to 0, \tag{2.6}$$

and hence $a_i+a_{i-1}+C_i^n=p^{|\boldsymbol{m}|}C_i^n$, $i=0, 1, \cdots, n$ (here we set $a_{-1}=0$, see [5, Lemma 3.2]). It follows that $a_i=(p^{|\boldsymbol{m}|}-1)\left(\sum_{j=0}^{i}(-1)^jC_j^n\right)$, i. e.,

$$a_i=(p^{|\boldsymbol{m}|}-1)C_i^{n-1},\ i=0, 1, \cdots, n. \tag{2.7}$$

To summarize, we have the following theorem.

Theorem 2.1. *Let* Char $F=p>3$, $L=W(n, \boldsymbol{m})$. *Then* (1) $\widetilde{V}_0(n, \boldsymbol{m})$ *is reducible if and only if* $V_0=V_0(\lambda_i)$, $i=0, 1, \cdots, n$; (2) $\widetilde{M}(\lambda_0, n, \boldsymbol{m})=Z$, $\widetilde{M}(\lambda_i, n, \boldsymbol{m})=d_{i-1}\widetilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})$, $i=1, \cdots, n$; (3) $\dim\widetilde{M}(\lambda_0, n, \boldsymbol{m})=1$, $\dim\widetilde{M}(\lambda_i, n, \boldsymbol{m})=a_{i-1}$, $i=1, \cdots, n$; (4) The composition factors of $\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})$ *are* $\widetilde{M}(\lambda_i, n, \boldsymbol{m})$, $\widetilde{M}(\lambda_{i+1}, n, \boldsymbol{m})$ *and* Z $(C_i^n-\delta_{i0}$ *times*$)$, $i=0, 1, \cdots, n$.

Next, we shall consider the case $L=S(n, \boldsymbol{m})$. Since L is a subalgebra of $W(n, \boldsymbol{m})$, the minimal submdoule $\widetilde{M}:=\widetilde{M}(\lambda_i, n, \boldsymbol{m})\subseteq U:=d_{i-1}\widetilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})$. Direct computation shows $(\widetilde{M})_1=(\mathscr{U}(L)(1\otimes V_0(\lambda_i))_1=U_1$. Hence $l(U/M)<N(\boldsymbol{m})-2$ since $l(U)=N(\boldsymbol{m})-1$. It follows that U/M is trivial. The top space $U_{N(\boldsymbol{m})-1}$ is isomorphic to the top space of $\widetilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})$ which is L_0-irreducible. If $\widetilde{M}\neq U$, $l(\widetilde{M})<l(U)$ and the top space of $U/\widetilde{M}$ is isomorphic to $U_{N(\boldsymbol{m})-1}$. When $i>1$, $U_{N(\boldsymbol{m})-1}$ is not trivial and we have a contradiction. If $i=1$, $\widetilde{M}=\sum_{j\leqslant N(\boldsymbol{m})-2}U_j$ (cf. [5, Proposition 2.2]). We have the following theorem.

Theorem 2.2. *Let* Char $F=p>3$, $L=S(n, \boldsymbol{m})$ *and* $U(\lambda_i)=d_{i-1}\widetilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})$. *Then*

(1) $\widetilde{V}_0(n, \boldsymbol{m})$ *is reducible if and only if* $V_0=V_0(\lambda_i)$, $i=0, 1, \cdots, n-1$;

(2) $\widetilde{M}(\lambda_0, n, \boldsymbol{m})=Z$, $\widetilde{M}(\lambda_1, n, \boldsymbol{m})=\sum_{j=0}^{N(\boldsymbol{m})-2}U(\lambda_1)_j$, $\widetilde{M}(\lambda_i, n, \boldsymbol{m})=U(\lambda_i)$, $i=2$,

$\cdots, n-1$;

(3) $\dim \widetilde{M}(\lambda_0, n, \boldsymbol{m})=1$, $\dim \widetilde{M}(\lambda_i, n, \boldsymbol{m})=a_i-\delta_{i1}$, $i=1, \cdots, n-1$, *where* a_i *is defined by* (2.7);

(4) *the composition factors of* $\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})$ *are* $\widetilde{M}(\lambda_i, n, \boldsymbol{m})$, $\widetilde{M}(\lambda_{i+1}, n, \boldsymbol{m})$ *and* Z ($C_i^n-\delta_{i0}+\delta_{i1}$ *times*), $i=0, 1, \cdots, n-1$.

Now we consider the case $L=H(n, \boldsymbol{m})$ where $n=2r$ and $L_0=sp(n)$. Define linear transformation θ_1 of $V_0(\lambda_1)$:

$$\theta_1(dx_j)=\sigma(\tilde{j})dx_{\tilde{j}}, \ j=1, \cdots, n, \tag{2.8}$$

and extend it to a linear transformation θ_i of $V_0(\lambda_i)$:

$$\theta_i(dx_{j_1}\wedge\cdots\wedge dx_{j_i})=\theta_1(dx_{j_1})\wedge\cdots\wedge\theta_1(dx_{j_i}). \tag{2.9}$$

Then θ_i is an L_0-module isomorphism of $V_0(\lambda_i)$ onto $V_0(\lambda_i)^*$. (We identify $V_0(\lambda_i)$ with $V_0(\lambda_i)^*$ and $\{\omega_S\}$ with its dual basis.) If $S=\{j_1, \cdots, j_i\}$, set $\tilde{S}=\{\tilde{j}_1, \cdots, \tilde{j}_i\}$. By (2.9), we have

$$\theta\omega_S=\pm\omega_{\tilde{S}}. \tag{2.10}$$

Let $S^\wedge=I\backslash S$. Then $\omega_S\wedge\omega_{S^\wedge}=\eta(S)\omega_I$ where $\eta(S)=\pm1$. We obtain a non-degenerate pairing of $V_0(\lambda_i)$ and $V_0(\lambda_{n-i})$ by letting

$$(\omega_S, \omega_T)=\delta_{S^\wedge, T}\eta(S), \ \text{Card}\ S=i, \ \text{Card}\ T=n-i. \tag{6.11}$$

And we obtain an L_0-module isomorphism ψ_i of $V_0(\lambda_i)^*$ onto $V_0(\lambda_{n-i})$, where

$$\psi_i(\omega_S)=\eta(S)\omega_{S^\wedge}. \tag{2.12}$$

Set $\zeta_i=\psi_i\theta_i$. Then ζ_i is an L_0-module isomorphism of $V_0(\lambda_i)$ onto $V_0(\lambda_{n-i})$. We have

$$\zeta_i(\omega_S)=\pm\omega_{\tilde{S}^\wedge}, \tag{2.13}$$

where the sign is uniquely determined by (2.10) and (2.12). Extend ζ_i to a linear map of $\widetilde{V}_0(\lambda_i)$ onto $\widetilde{V}_0(\lambda_{n-i})$:

$$\zeta_i(f\omega_S)=f(\zeta_i(\omega_S)), \ f\in\mathfrak{A}(n). \tag{2.14}$$

Then ζ_i is an L-module isomorphism. Let

$$\bar{d}_i=\zeta_{n-i+1}d_{n-i}\zeta_i. \tag{2.15}$$

Then $\bar{d}_i$: $V_0(\lambda_i, n)\to\widetilde{V}_0(\lambda_{i-1}, n)$ is an L-module isomorphism. We have

$$\bar{d}_i(f\omega_S)=\sum_{j\in S}\pm(D_{\tilde{j}}f)\omega_{S\backslash\{j\}}, \tag{2.16}$$

where the signs are also uniquely determined. Obviously

$$\ker \bar{d}_i=\bar{d}_{i+1}\widetilde{V}_0(\lambda_{i+1}, n). \tag{2.17}$$

We have a series of L-modules:

$$d_{i-1}\bar{d}_i\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})\subset d_{i-1}\widetilde{V}_0(\lambda_{i-1}, n, \boldsymbol{m})\subset\widetilde{V}_0(\lambda_i, n, \boldsymbol{m}). \tag{2.18}$$

Proposition 2.2. $\bar{d}_{i+1}d_i\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})$ *is irreducible*, $i=1, \cdots, r$.

Proof Denote $U=\bar{d}_{i+1}d_i\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})$. We have $l(U)=N(\boldsymbol{m})-2$ and

$$U\cong d_i\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})/\ker \bar{d}_{i+1}\cap d_i\widetilde{V}_0(\lambda_i, n, \boldsymbol{m}). \tag{2.19}$$

The top space of U is isomorphic to the top space of $d_i\widetilde{V}_0(\lambda_i, n, \boldsymbol{m})$ which is L_0-irreducible. If U contains a proper submodule U', we may assume $U'=\mathscr{U}(L)$

$(1\otimes V_0(\lambda_i))$. Since $U'_0=U_0$, $l(U)<N(\boldsymbol{m})-2$ and U' is trivial (note that U is an $\bar{H}(n, \boldsymbol{m})$-module and apply [5, Lemma 2.6]), which is impossible.

Similar, we can prove $d_{i-1}\bar{d}_i\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$ to be irreducible. Hence $\bar{d}_{i+1}d_i\tilde{V}_0(\lambda_i, n, \boldsymbol{m})=d_{i-1}\bar{d}_i\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$. In fact, we have, *a fortiori*, the following lemma.

Lemma 2.2. $\bar{d}_{i+1}d_iv=d_{i-1}\bar{d}_iv$, $v\in\tilde{V}_0(\lambda_i, n)$.

Proof Suppose Char$F=p>0$. Choose $\boldsymbol{m}$ such that $v\in\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$. Let $S=\{1, 2, \cdots, i\}$. Direct computation shows $\bar{d}_{i+1}d_i(x^{(\pi)}\omega_S)=d_{i-1}\bar{d}_i(x^{(\pi)}\omega_S)$ (to check the signs of both sides is a quite tedious work!). The module $\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$ is generated by $x^{(\pi)}\omega_S$, and the conclusion follows. If Char $F=0$, suppose we have $\bar{d}_{i+1}d_i(x^{(\alpha)}\omega_T)\neq d_{i-1}\bar{d}_i(x^{(\alpha)}\omega_T)$ for some α and T. The coefficients occuring in both sides are integers. Take a sufficiently large prime p. By a moaulo p process we would obtain an inequality of characteristic p, which is contrary to what we have already proved.

Lemma 2.3. (1) $\bar{d}_{i+1}\tilde{V}_0(\lambda_{i+1}, n)\cap d_{i-1}\tilde{V}_0(\lambda_{i-1}, n)=\bar{d}_{i+1}d_i\tilde{V}_0(\lambda_i, n)$, $i=0, 1, \cdots, r$ (we set $\tilde{V}_0(\lambda_{-1}, n)=0$).

(2) $(\ker \bar{d}_{i+1})\cap(d_i\tilde{V}_0(\lambda_i, n, \boldsymbol{m}))=\bar{d}_{i+2}d_{i+1}\tilde{V}_0(\lambda_{i+1}, n, \boldsymbol{m})\otimes\langle d_ix^{(\pi_s)}\omega_S \mid \text{Card } S=i\rangle$, $i=0, 1, \cdots, r-1$.

Proof (1) By induction on i. (2) Analogous to Lemma 2.1.

Let $b_i=\dim\bar{d}_{i+1}d_i\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$, especially $b_0=0$. By (2.19), we have

$$b_i+b_{i+1}=a_i-C_i^n,\ i=0, 1, \cdots, r-1. \tag{2.20}$$

It follows that

$$b_i=(p^{|\boldsymbol{m}|}-1)C_{i-1}^{n-2}-C_{i-1}^{n-1},\ i=1, \cdots, r. \tag{2.21}$$

Theorem 2.3. *Let* Char $F=p>3$, $L=H(n, \boldsymbol{m})$, $n=2r$, *and* V_0 *be an irreducible* L_0*-module. Then*

(1) $V_0(n, \boldsymbol{m})$ *is reducible if and only if* $V_0=V_0(\lambda_i)$, $i=0, 1, \cdots r$;

(2) $\tilde{M}(\lambda_0, n, \boldsymbol{m})=Z$, $\tilde{M}(\lambda_i, n, \boldsymbol{m})=d_{i-1}\bar{d}_i\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$, $i=1, \cdots, r$;

(3) $\dim\tilde{M}(\lambda_0, n, \boldsymbol{m})=1$, $\dim\tilde{M}(\lambda_i, n, \boldsymbol{m})=b_i$, $i=1, \cdots, r$;

(4) *The composition factors of* $\tilde{V}_0(\lambda_i, n, \boldsymbol{m})$ *are* $\tilde{M}(\lambda_{i-1}, n, \boldsymbol{m})$ $(1-\delta_{i1}$ *times*$)$, $\tilde{M}(\lambda_i, n, \boldsymbol{m})$ (2 *times*), $\tilde{M}(\lambda_{i+1}, n, \boldsymbol{m})$ *and* $Z(C_i^n+C_i^{n+1}-2\delta_{i0}$ *times*$)$, $i=0, 1, \cdots, r$ $(\tilde{M}(\lambda_{-1}, n, \boldsymbol{m})=0)$.

Theorem 2.4. *Let* Char $F\neq 2, 3$, $L=W(n)$ *or* $S(n)$ *and* V_0 *be an irreducicle* L_0*-module,* $\dim V_0<\infty$. *Set* $n'=n$ *if* $L=W(n)$, $n'=n-1$ *if* $L=S(n)$. *Then*

(1) $\tilde{V}_0(n)$ *is reducible if and only if* $V_0=V_0(\lambda_i)$, $i=0, 1, \cdots, n-1$;

(2) $\tilde{M}(\lambda_0, n)=Z$, $\tilde{M}(\lambda_i, n)=d_{i-1}\tilde{V}_0(\lambda_{i-1}, n)$, $i=1, \cdots, n'$;

(3) *The composition factors of* $\tilde{V}_0(\lambda_i, n)$ *are* $\tilde{M}(\lambda_i, n)$ *and* $\tilde{M}(\lambda_{i+1}, n)$, $i=0, 1, \cdots, n'$. $(\tilde{M}(\lambda_{n+1}, n)=0)$.

Proof First let Char $F>3$.

(1) is obvious.

(2) If $i=0$, there is nothing to prove. Let $i>0$. By Theorems 2.1 and 2.2, for

every $v\in\bar{V}(\lambda_{i-1}, n)$, there is $\boldsymbol{m}$, m_i sufficiently large, such that $d_{i-1}v\in\widetilde{M}(\lambda_i, n, \boldsymbol{m}) =\mathfrak{A}(L(n, \boldsymbol{m}))(1\otimes V_0(\lambda_i))\subset\mathfrak{A}(L(n))(1\otimes V_0(\lambda_i))$. Hence $\widetilde{M}(\lambda_i, n)=d_{i-1}\bar{V}(\lambda_i, n)$.

(3) follows directly from (2.4). Now let Char $F=0$. Then $U_{\mathbb{Z}}=\langle d_{i-1}(x^{(\alpha)}\omega_T)\mid \alpha\in A(n)$, Card $T=i-1\rangle_{\mathbb{Z}}$ is a lattice of $d_{i-1}\bar{V}_0(\lambda_{i-1}, n)$. If $\widetilde{M}(\lambda_i, n)=u(L(n))(1\otimes V_0(\lambda_i))\neq d_{i-1}\widetilde{V}_0(\lambda_{i-1}, n)$, then $u(L(n))_{\mathbb{Z}}(\mathbf{1}\otimes V_0(\lambda_i)_{\mathbb{Z}})$ is a proper sublattice of $U_{\mathbb{Z}}$. Take a sufficiently large prime p such that V_{0K} is L_{0K}-irreducible, where K is an algebraically closed field of characteristic p. By a modulo p process we would have $U_K=(d_{i-1}\widetilde{V}_0(\lambda_{i-1}, n))_K$ reducible, a contradiction.

Similarly, we have the following theorem.

Theorem 2.5. *Let* Char $F\neq 2, 3$, $L=H(n)$, $n=2r$, *and* V_0 *be an irreducible* L_0*-module*, dim $V_0<\infty$. *Then*

(1) $\widetilde{V}_0(n)$ *is reducible if and only if* $V_0=V_0(\lambda_i)$, $i=0, 1, \cdots, r$;

(2) $\widetilde{M}(\lambda_i, n)=\bar{d}_{i+1}d_i\widetilde{V}_0(\lambda_i, n), =1, \cdots, r$, $\widetilde{M}(\lambda_0, n)=Z$;

(3) *The composition factors of* $\widetilde{V}_0(\lambda_i, n)$ *are* $\widetilde{M}(\lambda_{i-1}, n)$, $\widetilde{M}(\lambda_i, n)$ (2 times) *and* $\widetilde{M}(\lambda_{i+1}, n)$, $i=0, 1, \cdots, r$ $(\widetilde{M}(\lambda_{-1}, n)=0)$.

§3. Decomposition of $\underset{\sim}{V}{}_0^*(\lambda_i)$

When $V_0=V_0(\lambda_i)^*$, write $\underset{\sim}{V}_0=\underset{\sim}{V}{}_0^*(\lambda_i)$ and write $\underset{\sim}{M}(\lambda_i)$ to denote the irreducible negative graded module with top space $V_0(\lambda_i)^*$. (If it is necessary to distinguish infinite-dimensional and finite-dimensional cases, we shall write $\underset{\sim}{V}{}_0^*(\lambda_i, n)$, $\underset{\sim}{M}(\lambda_i, n)$ and $\underset{\sim}{V}{}_0^*(\lambda_i, n, \boldsymbol{m})$, $M(\lambda_i, n, \boldsymbol{m})$ respectively.)

Let (,) be the non-degenerate pairing of $\widetilde{V}_0(\lambda_i)$ and $\underset{\sim}{V}_0(\lambda_i)$ ([5, (3.2)]). Define L-module homomorphism d_i^*: $(\lambda_i)\to\underset{\sim}{V}{}_0^*(\lambda_{i-1})$ by

$$(d_{i-1}v, v^*)=(v, d_i^*v^*),\ v\in\widetilde{V}_0(\lambda_{i-1}),\ v^*\in\underset{\sim}{V}{}_0^*(\lambda_i). \tag{3.1}$$

Let $\{ty_{j_1}\wedge\cdots\wedge ty_{j_i}\}$ be the basis of $V_0(\lambda_i)^*$ dual to the basis $\{dx_{j_1}\wedge\cdots\wedge dx_{j_i}\}$ of $V_0(\lambda_i)$. Then

$$d_i^*(y^\beta ty_{j_1}\wedge\cdots\wedge ty_{j_i})=\sum_{k=1}^{i}(-1)^k y^{\beta+\varepsilon_{j_k}}ty_{j_1}\wedge\cdots\wedge ty_{j_{k-1}}\wedge ty_{j_{k+1}}\wedge\cdots\wedge ty_{j_i}. \tag{3.2}$$

When $L=H(n)$, we can define $\bar{d}_i^*$: $\underset{\sim}{V}{}_0^*(\lambda_i)\to\underset{\sim}{V}{}_0^*(\lambda_{i+1})$ analogous to $\bar{d}_i$. Using d_i^* (and $\bar{d}_i^*$ when L is of type H), we obtain the decomposition of $\underset{\sim}{V}{}_0^*(\lambda_i, n, \boldsymbol{m})$. Since $\underset{\sim}{V}{}_0^*(\lambda_i, n, \boldsymbol{m})\cong\widetilde{V}_0(\lambda_{n-i}, n, \boldsymbol{m})$ by (1.12) and [5, Lemma 2.4], we need not go through the details.

Remark 3.1. Ermolaev[2] stated some results about $\underset{\sim}{V}_0(n, \boldsymbol{m})$ for $L=W(n, \boldsymbol{m})$ and $S(n, \boldsymbol{m})$ (mostly without proofs). The ideas, methods and notations used in [2] are quite different from ours.

For infinite-dimensional cases $L=L(n)$, we have the following theorem.

Theorem 3.1. *Let* Char $F\neq 2, 3$, $L=W(n)$ *or* $S(n)$ *and* V_0 *be an irreducible*

L_0-module, $\dim V_0<\infty$. *Then*

(1) $\underset{\sim}{V}_0(n)$ *is reducible if and only if* $V_0=V_0(\lambda_i)^*$, $i=0, 1, \cdots, n-1$;

(2) $\underset{\sim}{M}(\lambda_0, n)=Z$, $\underset{\sim}{M}(\lambda_i, n)=d_i^*\underset{\sim}{V}_0^*(\lambda_i, n)$, $i=1, \cdots, n-1$;

(3) *the composition factors of* $\underset{\sim}{V}_0^*(\lambda_i, n)$ *are* $\underset{\sim}{M}(\lambda_{i+1}, n)$ *and* $\underset{\sim}{M}(\lambda_i, n)$, $i=0, 1, \cdots, n-1$.

Theorem 3.2. *Let* Char $F\neq 2, 3$, $L=H(n)$, $n=2r$ *and* V_0 *be an irreducible* L_0*-module*, $\dim V_0<\infty$. *Then*

(1) $\underset{\sim}{V}_0(n)$ *is reducible if and only if* $V_0=V_0(\lambda_i)^*$, $i=0, 1, \cdots, r$;

(2) $\underset{\sim}{M}(\lambda_0, n)=Z$, $\underset{\sim}{M}(\lambda_i, n)=\bar{d}_{i-1}^*d_i^*\underset{\sim}{V}_0^*(\lambda_i, n)$, $i=1, \cdots, r$;

(3) *the composition factors of* $\underset{\sim}{V}_0^*(\lambda_i, n)$ *are* $\underset{\sim}{M}(\lambda_{i-1}, n)$, $\underset{\sim}{M}(\lambda_i, n)$ (2 *times*) *and* $\underset{\sim}{M}(\lambda_{i+1}, n)$, $i=0, 1, \cdots, r$ $(\underset{\sim}{M}(\lambda_{-1}, n)=0)$.

§4. Filtered Modules

Let $\mathscr{L}=\bigotimes_{i\in\mathbb{Z}}\mathscr{L}_i$ be a graded Lie algebra.

Definition 4.1. (1) *An $\mathscr{L}$-module V is a negative filtered module if there is a filtration* $0=V_{(-1)}\subset V_{(0)}\subset V_{(1)}\subset\cdots$, $\bigcup_i V_{(i)}=V$, *such that* $\mathscr{L}_jV_{(i)}\subset V_{(i-j)}$. *The* $\mathscr{L}_0$*-module* V_0 *is called the top space of* V.

(2) *An $\mathscr{L}$-module V is a positive filtered module if there is a filtration* $V=V_{[-1]}\supset V_{[0]}\supset V_{[1]}\supset\cdots$, $\bigcap_i V_{[i]}=\{0\}$, *such that* $\mathscr{L}_jV_{[i]}\subset V_{[i+j]}$. *The* $\mathscr{L}_0$*-module* $V_{(-1)}/V_{[0]}$ *is called the base space of* V.

Note 4.1. If V is finite-dimensional, the concepts of negative and positive filtered modules are equivalent.

Note 4.2. The graded module associated with a negative (positive) filtered module is a negative (positive) graded module.

Let the $\mathscr{L}$-modules U and V be dual with respect to the non-degenerate pairing (,). If V is a negative (positive) filtered module, then by letting $U_{[i]}=V_{(i)}^{\perp}$ $(U_{(i)}=V_{[i]}^{\perp})$, U is a positive (negative) filtered module.

Lemma 4.1. *If V is an irreducible $\mathscr{L}$-module, then V has a negative filtered module constructure if and only if the highest space V^0 of V is non-zero. Moreover, if V^0 contains a minimal non-zero $\mathscr{L}_0$-submodule, then we may suppose the top space to be $\mathscr{L}_0$-irreducible.*

Proof Necessity is clear. For sufficiency, take a non-zero $\mathscr{L}_0$-submodule V_0 ($\mathscr{L}_0$-irreducible if possible) of V^0. Then $V=\mathscr{U}(\mathscr{L})V_0=\mathscr{U}(\mathscr{L}^-)V_0$. Let

$$V_{(i)}=(\sum_{j\leq i}(\mathscr{U}(\mathscr{L}^-)_{-j})V_0.$$

We obtain a negative filtered module structure with V_0 as top space.

Corollary 4.1. *If V has a negative filtered module structure*, $\dim V_{(0)}<\infty$

(especially if $\dim V<\infty$*), then it is always possible to suppose* $V_{(0)}$ *to be* $\mathscr{L}_0$*-irreducible.*

Lemma 4.2. *Let* V *be a negative or positive filtered module of* $\mathscr{L}$*,* G *the graded module associated with* V*. If* G *is irreducible, then* V *is irreducible.*

Proof Let V' be a proper submodule of V. Then V' inherits a filtered module structure from V. It is easily seen that the graded module associated with V' is isomorphic to a proper submodule of G.

Lemma 4.3. *Let* $L=L(n, \boldsymbol{m})$*,* ρ *be an irreducible representation in the module* V*. Then* $\rho(D_i)^{p^{m_i}}=c_i\mathbf{1}_V$*,* $c_i\in F$*, and* $\mathbf{c}=(c_1, \cdots, c_n)$ *is called the index of* ρ *or* V*.*

Proof We have $(\mathrm{ad}D_i)^{p^{m_i}}=0$. For any $x\in L$, $[\rho(D_i)^{p^{m_i}}, \rho(x)]=\rho((\mathrm{ad}\,D_i)^{p^{m_i}}\cdot x)=0$. The conclusion follows from Schur's Lemma.

Consider $\underset{\sim}{V}_0(n)\cong\mathscr{U}(L)\otimes_{n(B)}V_0$ ([5, Theorem 3.1]). Let $\mathbf{c}=(c_1, \cdots, c_n)$, $c_i\in F$, and I_c be the ideal of $\mathfrak{B}(n)$ generated by $\{y_i^{p^{m_i}}-c_i,\ i=1, \cdots, n\}$. It can be easily verified that I_c is an admissible $L(n, \boldsymbol{m})$-submodule of $\mathfrak{B}(n)$. Hence $I_c\rtimes V_0$ is an $L(n, \boldsymbol{m})$-submodule of $\underset{\sim}{V}_0(n)$. Denote $\underset{\sim}{V}_0(n, \boldsymbol{m}, \mathbf{c})=\underset{\sim}{V}_0(n)/I_c\rtimes V_0$ and $\overline{\mathfrak{B}}_c=\mathfrak{B}(n)/I_c$. Then $\overline{\mathfrak{B}}_c=F[\bar{y}_1, \cdots, \bar{y}_n]$, $\bar{y}_i^{p^{m_i}}=c_i$, where $\bar{y}_i=y_i+I_c$. The Tricomi operator representation of $L(n, \boldsymbol{m})$ in $\mathfrak{B}(n)$ induces a representation in $\overline{\mathfrak{B}}_c$ and $\underset{\sim}{V}_0(n, \boldsymbol{m}, \mathbf{c})\cong\overline{\mathfrak{B}}_c\rtimes\underset{\sim}{V}_0$. $\underset{\sim}{V}_0(n, \boldsymbol{m}, \mathbf{c})$ inherits from $\underset{\sim}{V}_0(n)$ a negative filtered module structure. It is easy to see that its top space is isomorphic to V_0 and its associated graded module is isomorphic to $V_0(n, \boldsymbol{m})$. Obviously, the action of $D_i^{p^{m_i}}$ on $\underset{\sim}{V}_0(n, \boldsymbol{m}, \mathbf{c})$ is equal to $c_i\mathbf{1}$.

Denote $\underset{\sim}{V}_0(n, \boldsymbol{m}, \mathbf{c})$ by $\underset{\sim}{V}_0^*(\lambda_i, n, \boldsymbol{m}, \mathbf{c})$ when $V_0=V_0(\lambda_i)^*$. We have $d_i^*(I_c\rtimes V_0(\lambda_i)^*)\subset I_c\rtimes V_0(\lambda_{i-1})^*$ by (3.2). Hence d_i^* induces a module homomorphism $d_{i,c}^*$: $\underset{\sim}{V}_0^*(\lambda_i, n, \boldsymbol{m}, \mathbf{c})\to\underset{\sim}{V}_0^*(\lambda_{i-1}, n, \boldsymbol{m}, \mathbf{c})$. When $L=H(n, \boldsymbol{m})$, define $\bar{d}_{i,c}^*$ analogously.

Theorem 4.1. *Suppose* Char $F=p>3$, $L=W(n, \boldsymbol{m})$, $S(n, \boldsymbol{m})$ *or* $H(n, \boldsymbol{m})$. *Then*

(1) *for any given irreducible* L_0*-module* V_0 *and* $\mathbf{c}=(c_1, \cdots, c_n)$*, there exists, up to isomorphism, an unique irreducible negative filtered* L*-module* $\underset{\sim}{M}(V_0, \mathbf{c})$ *with top space* V_0 *and index* $\mathbf{c}$*; the map* $(V_0, \mathbf{c})\mapsto\underset{\sim}{M}(V_0, \mathbf{c})$ *is bijective;*

(2) $\underset{\sim}{M}(V_0, \mathbf{c})\cong\underset{\sim}{V}_0(n, \boldsymbol{m}, \mathbf{c})$ *if* $V_0\neq V_0(\lambda_i)^*$ *for any* i*;*

(3) *when* $V_0=V_0^*(\lambda_i)$*, using* $d_{i,c}^*$ *(and* $\bar{d}_{i,c}^*$ *if* $L=H(n, \boldsymbol{m})$*),* $\underset{\sim}{M}(V_0, \mathbf{c})$ *can be determined anabogous to the negative graded modules.*

Proof (2) The irreducibility of $\underset{\sim}{V}_0(n, \boldsymbol{m}, \mathbf{c})$ follows from Lemma 4.2.

(3) The irreducible negative filtered module $\underset{\sim}{M}(V_0(\lambda_i)^*, \mathbf{c})$ can be obtained by a discussion similar to that of $\underset{\sim}{M}(\lambda_i, n, \boldsymbol{m})$.

(1) Existence is clear. Let V be an irreducible negative filtered module with top space V_0 and index $\mathbf{c}$. By [5, Lemma 3.1], there is a surjective homomorphism φ: $\underset{\sim}{V}_0(n)\to V$ and $\ker\varphi\supset I_c\rtimes V_0$. Hence V is a uniquely determined homomorphic image of $\underset{\sim}{V}_0(n, \boldsymbol{m}, \mathbf{c})$. Conversely, for every $\underset{\sim}{M}(V_0, \mathbf{c})$ we can show that its highest

space is isomorphic to V_0. Thus, $\underset{\sim}{M}(V_0, c)$ is not isomorphic to $\underset{\sim}{M}(V'_0, \mathbf{c}')$ if $(V_0, \mathbf{c}) \neq (V'_0, \mathbf{c}')$ and the bijectivity is proved.

Theorem 4.2. *Let* Char $F \neq 2$, 3, $L=W(n)$, $S(n)$ *or* $H(n)$ *and* V_0 *be an finite-dimensional irreducible* L_0*-module. Then there exists, up to isomorphism, a unique irreducible negative filtered* L*-module* V *with top space* V_0 *and* V *is isomorphic to the negative graded module* $\underset{\sim}{M}(V_0)$.

Proof V is a homomorphic image of $\underset{\sim}{V}_0(n)$. Investigating the composition factors of $\underset{\sim}{V}_0(n)$, we come to our conclusion.

Now we consider the positive filtered modules of $L=L(n, \boldsymbol{m})$ and proceed to construct all irreducible positive filtered L-modules explicitly.

Let V_0 be an irreducible L_0-module, $\mathbf{c}=(c_1, \cdots, c_n)$ and

$$e(\mathbf{c}) = \exp\Big(\sum_{k=1}^{n} c_i x_i\Big) := \sum_{j=0}^{\infty}\Big(\sum_{k=1}^{n} c_i x_i\Big)^{(j)}$$

which is an element of the dual space $\mathfrak{B}^*$ of $\mathfrak{B}$. Set $\mathfrak{A}_c(n, \boldsymbol{m}) = \langle x^{(\alpha)}e(\mathbf{c}) \mid \alpha \in A(n, \boldsymbol{m})\rangle$. We can define the derivation representation of L in $\mathfrak{A}_c(n, \boldsymbol{m})$ to make it an admissible L-module. It is not difficult to verify that $(\mathfrak{B}_c(n, \boldsymbol{m}), I_c) = 0$ in the natural pairing (,) of $\mathfrak{B}^*$ and $\mathfrak{B}$. Hence the elements of $\mathfrak{A}_c(n, \boldsymbol{m})$ can be identified with the linear functions on $\mathfrak{B}_c$. Comparing dimensions, we have $\mathfrak{A}_c(n, \boldsymbol{m}) \cong \mathfrak{B}_c^*$. Let $\widetilde{V}_0(n, \boldsymbol{m}, \mathbf{c}) = \mathfrak{A}_c(n, \boldsymbol{m}) \rtimes V_0$. Then $\widetilde{V}_0(n, \boldsymbol{m}, \mathbf{c}) \cong \underset{\sim}{V}_0^*(n, \boldsymbol{m}, \mathbf{c})^*$. From the negative filtered module constructure of $V_0^*(n, \boldsymbol{m}, \mathbf{c})$ we obtain the positive filtered module constructure of $\widetilde{V}_0(n, \boldsymbol{m}, \mathbf{c})$ which has V_0 as base space and $\mathbf{c}$ as index. If $V_0 \neq V_0(\lambda_i)$, $\widetilde{V}_0(n, \boldsymbol{m}, \mathbf{c})$ is irreducible. If $V_0 = V_0(\lambda_i)$, an "exterior differential operator" $d_{i,c}$ (as well as $\bar{d}_{i,c}$ when $L=H(n, \boldsymbol{m})$) can be defined, by which all irreducible positive filtered modules $\widetilde{M}(V_0, \mathbf{c})$ are obtained.

Let $\mathfrak{A}_c(n) = \langle x^{(\alpha)}e(\mathbf{c}) \mid \alpha \in A(n)\rangle$ and $\widetilde{V}_0(n, \mathbf{c}) = \mathfrak{A}_c(n) \rtimes V_0$. Then $\widetilde{V}_0(n, \mathbf{c})$ is a positive filtered $L(n)$-module. If V_0 is L_0-irreducible, an irreducible positive filtered module $\widetilde{M}(V_0, \mathbf{c})$ with base space V_0 can be derived from $\widetilde{V}_0(n, \mathbf{c})$. If $\mathbf{c} \neq (0, \cdots, 0)$, $\widetilde{M}(V_0, \mathbf{c})$ is not graded. Thus, in the infinite-dimensional case $L=L(n)$, the situations of positive and negative filtered modules are quite different.

The problem of determining all irreducible positive filtered $L(n)$-modules remains open.

Remark 4.1. All the results of the present article are valid without the assumption that F is algebraically closed. In fact, algebraic closedness is used only in Lemma 1.1 and Propositions 1.1 and 1.2, of which the essential point is to prove $(\mathfrak{A}(L)(1\otimes V_0)) \cap (\widetilde{V}_0)_{N(m)} \neq 0$. But this is a fact unaltered if F is extended to its algebraic closure K. Note that, for an irreducible L_0-module V_0, $(V_0)_K$ contains a submodule isomorphic to $V_0(\lambda_i)_K$ if and only if $V_0 \cong V_0(\lambda_i)$ over F. In zero characteristic case also apply [5, Corollary 2.5].

References

[1] Ermolaev, Yu. B., On irreducible modules with filtration ($p>0$), *Izv. VUZ, Matematika*, **25**: 7(1981), 80—84.

[2] Jacobson, N., Lie Algebras, Interscience Publishers (1962).

[3] Kostrikin, A. I., and Safarevic, I. R., Graded Lie algebras of finite characteristic, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **33**(1969), 251—322.

[4] Shen Guangyu, Graded modules of graded Lie algebras of Cartan type (I)-mixed product of modules *Scientia Sinica*(Ser. A), **29**: 6(1986), 570—581.

[5] Shen Guangyu, Graded modules of graded Lie algebras of Cartan type (II)-positive and negative graded modules *Scientia Sinica*(Ser. A), **29**: 10(1986), 1009—1019.

[6] Wan Zhexian, Lie Algebra, Science Publishing House, Beijing, 1964.

[7] Wilson, R. L., A structural characterization of the simple Lie algebras of generalized Cartan type over fields of prime characteristic, *J. Algebra*, **40**(1976), 418—465.