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INVERSE LIMITS OF AFFINE GROUP SCHEMES

WANG JIANPAN (王建磐)*

Abstract

This paper considers inverse systems of affine group schemes. The author establishes the existence of the limit of such a system and proves some properties of the limit—some about its structure and some about its representations and cohomologies. In particular, a new explanation of generic cohomology is obtained: Let $\widetilde{G}$ be the inverse limit of the following inverse system

$$G \xleftarrow{F} G \xleftarrow{F} G \xleftarrow{F} \cdots$$

where F is a Frobenius morphism of a linear algebraic group G. Then the generic cohomology of G (with respect to F) with coefficients in a rational G-module V is simply the rational cohomology of $\widetilde{G}$ with coefficients in V.

In [5] we proved that the generic cohomology is the cohomology in the category of so-called quasi-rational modules. Later S. Donkin [2] pointed out that the category of quasi-rational modules for a linear algebraic group is just the category of rational modules for another affine group scheme. We reconstructed this affine group scheme as an inverse limit and we found that the procedure of constructing the affine group scheme may be generalized. The present paper is a generalization. We consider an inverse system of affine group schemes. We construct its inverse limit and prove some properties of the limit–some about its structure and some about its representations and cohomologies. We reobtain some known concepts and results (with a little generalization) as a special case of our main results.

This paper consists of 3 sections. In Section 1 we establish the existence of inverse limits and prove some structure properties of the limits. Section 2 is devoted to a discussion on the representation and cohomology properties of the limits. In Section 3 we give some remarks on generic cohomology.

§1. Inverse Limits of Affine Group Schemes

In this section K is a fixed commutative ring (with 1) and all affine group

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* Department of Mathematics, East China Normal University, Shanghai, China.

schemes we consider are over K. The category of affine group schemes over K is denoted by $\mathscr{A}$.

Let S be a directed set, i.e. a partially ordered set with the property that for each pair r, s in S there exists $t \in S$ such that $r \leqslant t$ and $s \leqslant t$. An inverse system over $\mathscr{A}$ indexed by S consists of an affine group scheme G^r for each $r \in S$ and a morphism μ^{rs}: $G^r \to G^s$ for each pair $s \leqslant r$ in S, subject to the following conditions:

i) μ^{rr} is the identity morphism of G^r for each $r \in S$; and

ii) $\mu^{rt} = \mu^{st} \circ \mu^{rs}$ whenever $t \leqslant s \leqslant r$ in S.

A morphism of inverse system (G^r, μ^{rs}) to inverse system (H^r, ρ^{rs}) is a family (φ^r), where φ^r: $G^r \to H^r$ is a morphism for each $r \in S$ and $\rho^{rs} \circ \varphi^r = \varphi^s \circ \mu^{rs}$ whenever $s \leqslant r$ in S. Thus, inverse systems over $\mathscr{A}$ indexed by S and morphisms between them form a category which will be denoted by $\mathscr{A}^S$.

As an example of such inverse systems, we can fix an affine group scheme G, and then let $G^r = G$ for each $r \in S$ and let μ^{rs}: $G^r \to G^s$ be the identity morphism of G for each pair $s \leqslant r$ in S. The inverse system we obtain in this way is called the constant system associated with G and will be denoted by const(G). Obviously const is a functor from $\mathscr{A}$ to $\mathscr{A}^S$.

It is known that the right adjoint functor of const (if it does exist) is the inverse limit functor $\varprojlim$: $\mathscr{A}^S \to \mathscr{A}$. For a system (G^r, μ^{rs}) in $\mathscr{A}^S$, the inverse limit $G = \varprojlim G^r$ with the adjunction μ^r: $G \to G^r$ may be characterized by the following universal property: Suppose we are given an affine group scheme H and a morphism θ^r: $H \to G^r$ for each $r \in S$ subject to $\mu^{rs} \circ \theta^r = \theta^s$ whenever $s \leqslant r$ in S, then there exists a unique morphism θ: $H \to G$ such that $\mu^r \circ \theta = \theta^r$.

First we establish the existence of the functor $\varprojlim$:

Proposition 1.1. *Let (G^r, μ^{rs}) be an inverse system over $\mathscr{A}$ indexed by S. Then the inverse limit $\varprojlim G^r$ exists.*

Proof The dual category of $\mathscr{A}$ is the category of commutative K-Hopf algebras, so we only need to prove that a direct system of commutative K-Hopf algebras has a direct limit. It is well-known that a direct system of commutative K-algebras has a direct limit, and it is easy to define a comultiplication, a counit and an antipode to make the limit a K-Hopf algebra. It is also easy to prove this K-Hopf algebra to be the direct limit of the direct system in the category of commutative K-Hopf algebras. The details are left to the readers.

For a morphism (φ^r): $(G^r, \mu^{rs}) \to (H^r, \rho^{rs})$, we can obtain a morphism $\varprojlim \varphi^r$: $\varprojlim G^r \to \varprojlim H^r$ as follows: first we use the adjunction μ^s: $\varprojlim G^r \to G^s$ to obtain a morphism $\varphi^s \circ \mu^s$: $\varprojlim G^r \to H^s$, which commutes with ρ^{rs}, so we can further use the universal property of $\varprojlim H^r$ to obtain a morphism $\varprojlim \varphi^r$: $\varprojlim G^r \to \varprojlim H^r$. It is easy to verify that we have made $\varprojlim$ a functor from $\mathscr{A}^S$ to $\mathscr{A}$.

Next we let (G^r, μ^{rs}) be as before, and $A_r = K[G^r]$, the affine algebra of G^r. Let also μ_{sr} be the comorphism of μ^{rs}. Then (A_r, μ_{sr}) is a direct system of commutative K-Hopf algebras indexed by S. If $A = \varinjlim A_r$ and $G = \varprojlim G^r$, then $A \overset{\epsilon}{=} K[G]$. Therefore, if we want to know the srtuoture of G, we have to investigate the K-algebra homomorphisms of A to any commutative K-algebra. Because A is also the direct limit of direct system (A_r, μ_{sr}) in the category of commntative K-algebras, the K-algebra homomorphisms of A to a commntative K-algebra R are in one-one correspondence with the families (x_r), where $x_r: A_r \to R$ is a K-algebra homomorrphism with $x_r \circ \mu_{sr} = x_s$ for $s \leqslant r$ in S. It follows that the elements of $G(R)$ are in one-one correspondence with the families (x_r), where $x_r \in G^r(R)$ and $\mu^{rs}(x_r) = x_s$ whenever $s \leqslant r$. Thus, we have proved the following proposition.

Proposition 1.2. *Let (G^r, μ^{rs}) be an inverse system over $\mathscr{A}$ indexed by S. Then we have*

$$(\varprojlim G^r)(R) \cong \varprojlim (G^r(R))$$

for any commutative K-algebra R.

Now the adjunction μ^s: $\varprojlim G^r \to G^s$ and the morphism $\varprojlim \varphi^r$: $\varprojlim G^r \to \varprojlim H^r$, where (φ^r): $(G^r, \mu^{rs}) \to (H^r, \rho^{rs})$ is a morphism in $\mathscr{A}^S$, have concrete forms: μ^s is simply the projection to the s-coordinate, and $\varprojlim \varphi^r$ is simply the application of φ^r to the r-coordinate for each $r \in S$.

Next we shall prove that $\varprojlim$ also commutes with ker. The precise statement is as follows.

Theorem 1.3. *Let (φ^r): $(G^r, \mu^{rs}) \to (H^r, \rho^{rs})$ be a morphism in $\mathscr{A}^S$. Let $\varkappa^{rs} = \mu^{rs}|_{\ker\varphi^r}$. Then* $(\ker\mu^r, \varkappa^{rs})$ *is also an inverse system over $\mathscr{A}$ indexed by S, and*

$$\ker(\varprojlim \varphi^r) \cong \varprojlim (\ker \varphi^r).$$

Proof Let $Q^r = \ker \varphi^r$ and ν^r: $Q^r \to G^r$ be the canonical embedding. Obviously (ν^r) is a morphism of $(Q^r, \varkappa^{rs})$ to (G^r, μ^{rs}). Being a right adjoint, $\varprojlim$ preserves kernels, so it is enough to prove that (ν^r) is the kernel of (φ^r) in $\mathscr{A}^S$. On the one hand, we have $(\varphi^r) \circ (\nu^r) = (\varphi^r \circ \nu^r) = 0$. On the other hand, if we are given a morphism (ψ^r): $(P^r, \xi^{rs}) \to (G^r, \mu^{rs})$ with $(\varphi^r) \circ (\psi^r) = (\varphi^r \circ \psi^r) = 0$, then for each $r \in S$ there exists a unique morphism ζ^r: $P^r \to Q^r$ such that $\psi^r = \nu^r \circ \zeta^r$, for ν^r is the kernel of φ^r in $\mathscr{A}$. We want to show (ζ^r) is a morphism of (P^r, ξ^{rs}) to $(Q^r, \varkappa^{rs})$, i. e. $\zeta^s \circ \xi^{rs} = \varkappa^{rs} \circ \zeta^r$ whenever $s \leqslant r$ in S. In fact

$$\nu^s \circ \zeta^s \circ \xi^{rs} = \psi^s \circ \xi^{rs} = \mu^{rs} \circ \psi^r = \mu^{rs} \circ \nu^r \circ \zeta^r = \nu^s \circ \varkappa^{rs} \circ \zeta^r.$$

Since ν^s is a monomorphism, we deduce $\zeta^s \circ \xi^{rs} = \varkappa^{rs} \circ \zeta^r$, as required. Therefore, (ζ^r) is a morphism of (P^r, ξ^{rs}) to $(Q^r, \varkappa^{rs})$ with $(\psi^r) = (\nu^r) \circ (\zeta^r)$, and the uniqueness of such morphism follows from the uniqueness of ζ^r for each r. Hence (ν^r) is the kernel of (φ^r) in $\mathscr{A}^S$.

§ 2. Representations and Cohomologies of an Inverse Limit

In this section we assume for convenience that K is a field. We consider an inverse system (G^r, μ^{rs}) over $\mathscr{A}$ indexed by S again. We shall seek some relations between cohomologies of G and those of the limit $G = \varprojlim G^r$.

Lemma 2.1. *Let V be a finite dimensional rational G–module. Then there exists an index $r \in S$ such that V has a rational G^r–module structure which goes back to the original G–module structure via the adjunction μ^r: $G - G^r$.*

Proof Let $A = K[G]$, which is the direct limit of $A_r = K[G^r]$, as mentioned before. Let Δ_V: $V \to V \otimes A$ be the A–comodule structure map associated with the given G–module structure on V. Choose a K–basis of V, say $v_1, v_2, \cdots, v_n$, and let

$$\Delta_V(v_j) = \sum_{i=1}^{n} v_i \otimes a_{ij}, \quad a_{ij} \in A$$

for $j = 1, 2, \cdots, n$. Since $\{a_{ij}\}$ is a finite subset of A, we can find an index s and $b_{ij} \in A_s$ for $i, j = 1, 2, \cdots, n$ such that $\mu_s(b_{ij}) = a_{ij}$, μ_s being the comorphism of μ^s. Let B be the Hopf subalgebra of A_s generated by these b_{ij}. Then B is finitely generated as a K–algebra (see, for example, [1, Lemma 3.4.5]), so it is a noetherian ring. Let $I = \ker \mu_s$ and $J = B \cap I$. Being an ideal of a noetherian ring, J is finitely generated, so we can find an index $r \geqslant s$ such that $\mu_{sr}(J) = 0$, μ_{sr} being the comorphism of μ^{rs}. Now μ_r: $A_r \to A$ maps $B' = \mu_{sr}(B)$ isomorphically onto a Hopf subalgebra A' of A. Since A' contains all of the a_{ij}, V is an A'–comodule thus also a B'–comodule provided we let

$$\Delta'_V(v_j) = \sum_{i=1}^{n} v_i \otimes \mu_{sr}(b_{ij}).$$

But B' is a Hopf subalgebra of A_r, therefore V becomes an A_r–comodule under this definition. Hence V becomes a rational G^r–module, which obviously goes back to the original G–module via μ^r.

Now we consider the following situation: we are given a rational G^r–module M_r for each $r \in S$ and a G^r–homomorphism τ_{sr}: $M_s \to M_r$ for each pair $s \leqslant r$ in S, where M_s is considered as a G^r–module via μ^{rs}: $G^r \to G^s$, subject to

i) τ_{rr} is the identity homomorphism of M_r; and

ii) $\tau_{tr} = \tau_{sr} \circ \tau_{ts}$ whenever $t \leqslant s \leqslant r$ in S.

Obviously, if we define a G–module structure on each M_r via the adjunction μ^r: $G\to G^r$, then (M_r, τ_{sr}) becomes a direct system of rational G–modules indexed by S. Such a system will be called a direct system of rational G^r–modules. We can form the direct limit of the system as a sysem of rational G–modules: $M=\varinjlim M_r$ and we have a canonical G–homomorphism τ_r: $M_r\to M$.

Lemma 2.2. *Let (M_r, τ_{sr}) be a direct system of rational G^r–modules, and V a finite dimensional rational G–module. Then any G–homomorphism φ: $V\to M$ factors through τ_r for some $r\in S$. More precisely, there exists an index $r\in S$ such that*

i) *V can be given a G^r–module structure as in (2.1); and*

ii) *there exists a G^r–homomorphism φ_r: $V\to M_r$ with $\varphi=\tau_r\circ\varphi_r$.*

Proof Obviously there exists an index $s\in S$ such that $\varphi(V)\subset\tau_s(M_s)$ and V has a G^s–module structure as in (2.1). We choose a finite dimensional G^s–submodule U of M_s such that $\varphi(V)\subset\tau_s(U)$. If necessary we change a bigger index for s and can assume that τ_s maps U injectively into M. Now, as in the proof of (2.1), we can find a Hopf subalgebra B of A_s which is finitely generated as a K–algebra such that the A_s–comodule structures of V and U are realized in B. Then there exists an index $r\geqslant s$ in S such that the ideal $B\cap\ker\mu_s$ of B is killed by μ_{sr}, μ_s and μ_{sr} being the comorphisms of μ^s and μ^{rs}, respectively. Now $B'=\mu_{sr}(B)$ is a Hopf subalgebra of A and it may be regarded as a Hopf subalgebra of A, and V and $U'=\mu_{sr}(U)$ may be regarded as B'–comodules. Since φ is a G–homomorphism sending V into $\mu_s(U)=\mu_r(U')$, it is a B'–comodule homomorphism sending V into $\mu_r(U')$. But U' and $\mu_r(U')$ are isomorphic as B'–comodules, so we can define a B'–comodule homomorphism φ_r sending V into U' with $\varphi=\tau_r\circ\varphi_r$. Obviously φ_r can be regarded as an A_r–comodule homomorphism sending V into M_r, so we have done.

Proposition 2.3. *Let (I_r, τ_{sr}) be a direct system of G^r–modules with I_r a rationally injective G^r–module for each $r\in S$. Then $I=\varinjlim I_r$ is a rationally injective G–module.*

Proof It is enough to show that for any injective G–homomorphism φ: $M\to N$ with N finite dimensional and any G–homomorphism θ: $M\to I$ there exists a G–homomorphism $\tilde\theta$: $N\to I$ such that $\theta=\tilde\theta\circ\varphi$. Thanks to (2.1) and (2.2), there exists an index $r\in S$ such that

i) M and N have G^r–module structures as in (2.1);

ii) φ is a G^r–homomorphism;

iii) θ factors as $\theta=\tau_r\circ\psi_r$ with ψ_r: $M\to I_r$ a G^r–homomorphism.

Since I_r is a rationally injective G^r–module, we have a G^r–homomorphism $\tilde\psi_r$: $N\to I_r$ such that $\psi_r=\tilde\psi_r\circ\varphi$. Now let $\theta=\tau_r\circ\tilde\psi_r$. Then

$$\tilde{\theta}\circ\varphi=\tau_r\circ\tilde{\psi}_r\circ\varphi=\tau_r\circ\psi_r=\theta,$$

as required.

Next we state the main theorem.

Theorem 2.4. *Let (M_r, τ_{sr}) be a direct system of G^r-modules and V a finite dimensional G-module. We choose an index $r_0\in S$ such that V has a G^{r_0}-module structure as in (2.1), and define a G^{r_0}-module structure on V via μ^{rr_0} for each $r\geqslant r_0$ in S. Then, for r, $s\in S$ with $r_0\leqslant s\leqslant r$ and any $n\in\mathbb{Z}^+$, τ_{sr} induces a canonical homomorphism*

$$\operatorname{Ext}^n(\tau_{sr})\colon \operatorname{Ext}^n_{G^s}(V, M_s)\to\operatorname{Ext}^n_{G^r}(V, M_r).$$

These homomorphisms make $(\operatorname{Ext}^n_{G^r}(V, M_r), \operatorname{Ext}^n(\tau_{sr}))$ a direct system of K-linear spaces indexed by $\{r\in S\mid r\geqslant r_0\}$, and there exists a canonical and natural isomorphism

$$\varinjlim_{r\geqslant r_0}\operatorname{Ext}^n_{G^r}(V, M_r)\cong\operatorname{Ext}^n_G(V, M).$$

Proof We first prove the theorem for $n=0$. The existence of

$$\operatorname{Hom}(\tau_{sr})\colon \operatorname{Hom}_{G^s}(V, M_s)\to\operatorname{Hom}_{G^r}(V, M_r)$$

is a trivial fact, and obviously these homomorphisms make $(\operatorname{Hom}_{G^r}(V, M_r), \operatorname{Hom}(\tau_{sr}))$ a direct system. We can define a canonical and natural homomorphism

$$\theta\colon \varinjlim_{r\geqslant r_0}\operatorname{Hom}_{G^r}(V, M_r)\to\operatorname{Hom}_G(V, M)$$

by sending each representative $\alpha\in\operatorname{Hom}_{G^r}(V, M_r)$ to $\tau_r\circ\alpha$. The surjectivity of θ is, in fact, the conclusion of (2.2). To prove the injectivity we assume that there is a G^r-homomorphism $\alpha\colon V\to M$ for some $r\geqslant r_0$ with $\tau_r\circ\alpha=0$, i. e. $\tau_r(\alpha(V))=0$. Since $\alpha(V)$ is finite dimensional, there exists an index $s\geqslant r$ such that $\tau_{rs}(\alpha(V))=0$, i. e. $\tau_{rs}\circ\alpha=0$. It means that α is 0 as an element of $\varinjlim_{r\geqslant r_0}\operatorname{Hom}_{G^r}(V, M_r)$, which proves the theorem for $n=0$.

Next we choose a rationally injective G^s-resolution of M_s for each $s\geqslant r_0$:

$$0(\to M_s)\to I_s^0\to I_s^1\to I_s^2\to\cdots.$$

This resolution may be regarded as an acyclic complex of G^r-modules for each $r\geqslant s$ via μ^{rs}. Therefore we have a complex map extending the G^r-homomorphism τ_{sr}:

$$(*)\qquad \begin{array}{ccccccccc} 0(\longrightarrow & M_s) & \longrightarrow & I_s^0 & \longrightarrow & I_s^1 & \longrightarrow & I_s^2 & \longrightarrow\cdots \\ & \downarrow{\scriptstyle\tau_{sr}} & & \downarrow{\scriptstyle\tau_{sr}^0} & & \downarrow{\scriptstyle\tau_{sr}^1} & & \downarrow{\scriptstyle\tau_{sr}^2} & \\ 0(\longrightarrow & M_r) & \longrightarrow & I_r^0 & \longrightarrow & I_r^1 & \longrightarrow & I_r^2 & \longrightarrow\cdots \end{array}$$

We apply functor $\operatorname{Hom}_{G^r}(V, -)$ to the resolution of M_r for each r to get the following diagram for each pair $s\leqslant r$:

$$\begin{array}{ccccc} 0\longrightarrow & \operatorname{Hom}_{G^s}(V, I_s^0) & \longrightarrow & \operatorname{Hom}_{G^s}(V, I_s^1) & \longrightarrow\cdots \\ & \downarrow{\scriptstyle\operatorname{Hom}(\tau_{sr}^0)} & & \downarrow{\scriptstyle\operatorname{Hom}(\tau_{sr}^0)} & \\ 0\longrightarrow & \operatorname{Hom}_{G^r}(V, I_r^0) & \longrightarrow & \operatorname{Hom}_{G^r}(V, I_r^1) & \longrightarrow\cdots \end{array}$$

By taking cohomology, we obtain a homomorphism

$$\operatorname{Ext}^n(\tau_{sr})\colon \operatorname{Ext}^n_{G^s}(V, M_s)\to\operatorname{Ext}^n_{G^r}(V, M_r).$$

Thanks to the general theory of homological algebra, the homomorphisms we get in this way are independent of the choice of the resolutions and the choice of the complex maps, so they are canonical, and it is easy to see that they are also natural. It is clear also that the conditions for direct systems are satisfied by those homomorphisms. Therefore $(\mathrm{Ext}^n_{G^r}(V, M_r), \mathrm{Ext}^n(\tau_{sr}))$ becomes a direct system of K-linear spaces for each $n\in\mathbb{Z}^+$.

We go back to diagram (*) again. We can construct a special injective resolution for each M_r so that (I^n_r, τ^n_{sr}) is a direct system of G^r-modules for each $n\in\mathbb{Z}^+$. If $S=\mathbb{N}$, the natural numbers, this is easy. In general, this also can be done. For example, we can take the standard resolution

$$I^n_r = M_r\otimes\overbrace{K[G^r]\otimes\cdots\otimes K[G^r]}^{n+1\text{ copies}},$$

and

$$\tau^n_{sr} = \tau_{sr}\otimes\overbrace{\mu_{sr}\otimes\cdots\otimes\mu_{sr}}^{n+1\text{ factors}},$$

μ_{sr} being the comorphism of μ^{rs}. Now we arrive at a direct system of resolutions whose limit, by the fact that $\varinjlim$ preserves exactness and (2.3), is a rationally injective resolution of M. By applying functor $\mathrm{Hom}_{G^r}(V, -)$ to the resolution of M_r we obtain a direct system of complexes whose limit, by the theorem for $n=0$ we have proved, is the complex obtained by applying functor $\mathrm{Hom}_G(V, -)$ to the limit resolution. Finally, by repeatedly using the fact that $\varinjlim$ preserves exactness, we obtain the required canonical and natural isomorphism

$$\varinjlim_{r\geq r_0}\mathrm{Ext}^n_{G^r}(V, M_r)\cong\mathrm{Ext}^n_G(V, M).$$

Corollary 2.5. *Let (M_r, τ_{sr}) be a direct system of G^r-modules. Then for any $s\leqslant r$ in S and any $n\in\mathbb{Z}^+$, τ_{sr} induces a canonical homomorphism*

$$H^n(\tau_{sr}): H^n(G^s, M_s)\to H^n(G^r, M_r).$$

These homomorphisms make $(H^n(G^r, M_r), H^n(\tau_{sr}))$ a direct system of K-linear spaces indexed by S, and there exists a canonical and natural isomorphism

$$\varinjlim H^n(G^r, M_r)\cong H^n(G, M).$$

Proof Let $V=K$, the one-dimensional trivial module, in (2.4). Note that K may be regarded as a G^r-module for each $r\in S$, so the condition $r\geqslant r_0$ is unnecessary.

Recall that for a morphism of affine group schemes $\varphi: H\to G$ we can define an induction functor ind_φ from the category of rational H-modules to that of rational G-modules. This functor is the right adjoint of the functor res_φ which is simply viewing a rational G-module as a rational H-module via φ. Functor ind_φ is left exact. We are also interested in its right derived functors $R^n\,\mathrm{ind}_\varphi$.

Recall also that $\mathrm{ind}_\varphi V \cong (V \otimes K[G])^H$ and, more generally,

$$R^n \mathrm{ind}_\varphi V \cong H^n(H, V \otimes K[G]), \ n \in \mathbb{Z}^+.$$

Using this fact and noting that

$$\varinjlim_r (U_r \otimes V_r) \cong (\varinjlim_r U_r) \otimes (\varinjlim_r V_r)$$

for two direct systems of K-linear spaces indexed by S, we deduce immediately the following result.

Corollary 2.6. *Let (H^r, ρ^{rs}) and (G^r, μ^{rs}) be two inverse systems over $\mathscr{A}$ indexed by S and (φ^r): $(H^r, \rho^{rs}) \to (G^r, \mu^{rs})$ be a morphism. If we have a direct system of H^r-modules (M_r, τ_{sr}), then for any $s \leqslant r$ in S and any $n \in \mathbb{Z}^+$, τ_{sr} induces a canonical and natural G-homomorphism*

$$R^n \mathrm{ind}(\tau_{sr}): R^n \mathrm{ind}_{\varphi^s} M_s \to R^n \mathrm{ind}_{\varphi^r} M_r.$$

These homomorphisms make $(R^n \mathrm{ind}_{\varphi^r} M_r, R^n \mathrm{ind}(\tau_{sr}))$ a direct system of rational G^r-modules, and there exists a canonical and natural G-isomorphism

$$\varinjlim R^n \mathrm{ind}_{\varphi^r} M_r \cong R^n \mathrm{ind}_\varphi M,$$

where $\varphi = \varprojlim \varphi^r$ and $M = \varinjlim M_r$.

Proof Use (2.5) to the direct system of H^r-modules $(M_r \otimes K[G^r], \tau_{sr} \otimes \mu_{sr})$, μ_{sr} being the comorphism of μ^{rs}, and note that the procedure of taking limit is compatible with the actions of G^r.

If we replace the direct system (M_r, τ_{sr}) by a finite dimensional rational G-module (or a finite dimensional rational H-module, in (2.6)) M in (2.4)—(2.6), we can deduce the following results. The proofs of these results are almost the same: using (2.1) we regard M as a direct system of G^r (or H^r)-modules indexed by a subset of S, say $S_0 = \{r \in S \mid r \geqslant r_0\}$ for an index $r_0 \in S$, then we apply (2.4), or (2.5), or (2.6) with S replaced by S_0.

Corollary 2.7. *Let V and M be finite dimensional G-modules. We choose an index $r_0 \in S$ such that V and M have rational G^{r_0}-module structures as in (2.1), and define G^r-module structures on V and M via μ^{rr_0} for any $r \geqslant r_0$. Then there exists a canonical and natural isomorphism*

$$\varinjlim_{r \geqslant r_0} \mathrm{Ext}^n_{G^r}(V, M) \cong \mathrm{Ext}^n_G(V, M).$$

Corollary 2.8. *Let M be a finite dimesional G-module. We choose an index $r_0 \in S$ such that M has a rational G^{r_0}-module structure as in (2.1), and define a G^r-module structure on M via μ^{rr_0} for any $r \geqslant r_0$. Then there exists a canonical and natural isomorphism*

$$\varinjlim_{r \geqslant r_0} H^n(G^r, M) \cong H^n(G, M).$$

Corollary 2.9. *Let (φ^r): $(H^r, \rho^{rs}) \to (G^r, \mu^{rs})$ be as in (2.6), and M be a finite* *dimensional rational H-module. We choose an index $r_0\in S$ such that M has an rational H^{r_0}-module structure as in* (2.1), *and define an H^r-module structure on M via ρ^{rr_0} for any $r\geqslant r_0$. Then there exists a canonical and natural isomorphism*

$$\varinjlim_{r\geqslant r_0} R^n\,\mathrm{ind}_{\varphi^r} M\cong R^n\,\mathrm{ind}_{\varphi} M.$$

§3. Some Remarks on Generic Cohomology

In this section K is an algebraically closed field of characteristic $p>0$ and G is a linear algebraic group over K. Let F: $G\to G$ be a fixed Frobenius endomorphism of G. Then we can construct an inverse system indexed by the natural numbers $\mathbb{N}$

$$G^0\overset{\mu^{10}}{\longleftarrow}G^1\overset{\mu^{21}}{\longleftarrow}G^2\overset{\mu^{32}}{\longleftarrow}G^3\overset{\mu^{43}}{\longleftarrow}G^4\longleftarrow\cdots$$

with $G^i=G$ and $\mu^{i+1,i}=F$ for $i\in\mathbb{N}$. This system differs from const (G) defined in Section 1, for the morphisms involved here are not identities. Let $\widetilde{G}=\varprojlim G^i$.

Theorem 3.1. *Let V be a rational G-module. If we regard V as a rational $\widetilde{G}$-module via the adjunction μ^0: $\widetilde{G}\to G^0=G$, then there exists a canonical and natural isomorphism*

$$H^n_{\mathrm{gen}}(G, V)\cong H^n(\widetilde{G}, V),$$

for any $n\in\mathbb{Z}^+$.

Proof Obviously the G^i-module structure given via μ^{i0} is just $V^{(i)}$, the i-th Frobenius twist of V. Theretore (2.5) gives a canonical and natural isomorphism

$$\begin{aligned}H^n(\widetilde{G}, V)&\cong\varinjlim H^n(G^i, V)\\&=\varinjlim H^n(G, V^{(i)})\\&=H^n_{\mathrm{gen}}(G, V),\end{aligned}$$

by the definition of generic cohomology.

In [5] we defined generic cohomology as the cohomology of the category of quasi-rational G-modules. The quasi-rational G-modules are just rational $\widetilde{G}$-modules. In fact, (2.1) tells us that any finite dimensional rational $\widetilde{G}$-module is a rational G^i-module for large i. This means that any finite dimensional rational $\widetilde{G}$-module is a G-module which need not be rational but can be changed into a rational G-module by a power of F, and thus any rational $\widetilde{G}$-module is a direct limit of such finite dimensional G-modules. This is just the defining conditions of quasi-rational G-modules. From this we see that (3.1) is a version of [5, Th. 3.1].

It is easy to construct $K[\widetilde{G}]$ directly. Let A_0 be the set of K-valued functions defined on G. Then A_0 is a commutative K-Hopf algebra in the usual way, and there is a Frobenius endomorphism $F^\#$ defined on it:

$$F^\#(f)(x)=f(F(x)),\ \text{for all}\ f\in A_0,\ x\in G.$$

Now let

$$A=\{f\in A_0 \mid (F^{\#})^i(f)\in K[G] \text{ for some } i\in\mathbb{N}\}.$$

Then A is a Hopf subalgebra of A_0, so it defines an affine group scheme, which is just $\widetilde{G}$, i. e. $A=K[\widetilde{G}]$. The detail verification is left to the readers.

The adjunction μ^0: $\widetilde{G}\to G^0=G$ is an epimorphism, for $K[G]$ is a Hopf subalgebra of $K[\widetilde{G}]$, so we have an exact sequence of affine group schemes:

$$E\to Q\to \widetilde{G}\to G\to E,$$

where E is the trivial affine group scheme, and Q is the kernel of μ^0.

Proposition 3.2. $Q\cong\varprojlim G_i$, *where* $G_i=\ker F^i$, *an infinitesimal subgroup scheme of* G, *and the homomorphism* $G_j\to G_i$ *for* $i\leqslant j$ *is the restriction of* F^{j-i}: $G\to G$.

Proof We can define a morphism from (G^i, μ^{ji}) to cons t (G) as follows:

$$\begin{array}{ccccccccc}
G^0 & \xleftarrow{\mu^{10}} & G^1 & \xleftarrow{\mu^{21}} & G^2 & \xleftarrow{\mu^{32}} & G^3 & \xleftarrow{\mu^{43}} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F^2} & & \downarrow{\scriptstyle F^3} & & \\
G & \xleftarrow{\mathrm{id}} & G & \xleftarrow{\mathrm{id}} & G & \xleftarrow{\mathrm{id}} & G & \xleftarrow{\mathrm{id}} & \cdots
\end{array}$$

It is easy to see that the limit of this morphism is just the adjunction μ^0:$\widetilde{G}\to G^0=G$. So, by (1.3), the kernel of μ^0 is the inverse limit of the inverse system $(G_i, \varkappa^{ji})$, where $G_i=\ker F^i$ and $\varkappa^{ji}$: $G_j\to G_i$ is the restriction of μ^{ji}, which is just F^{j-i}.

The affine algebra of Q is A/MA, where M is the augmentation ideal of $K[G]$. This follows from the general theory of affine group schemes.

Because Q is a normal subgroup scheme of $\widetilde{G}$, we have a Lyndon–Hochschild–Serre spectral sequence for any rational $\widetilde{G}$–module V:

(3.3) $$E_2^{m,n}=H^m(G, H^n(Q, V))\Rightarrow H^{m+n}(\widetilde{G}, V).$$

If V is a rational G–module, then it is trivial as a rational Q–module, so the spectral sequence becomes

(3.4) $$E_2^{m,n}=H^m(G, H^n(Q, K)\otimes V)\Rightarrow H^{m+n}_{\mathrm{gen}}(G, V),$$

where K is the one-dimensional trivial Q–module. If we let $H^n=H^n(Q, K)$, it is known that

$$H=\coprod_{n\in\mathbb{Z}^+} H^n$$

has a graded K–algebra structure. It is the cohomology ring of Q. Of course H is also a rational G–module, so we shall call it a rational G–algebra. From (3.4) we see that H will play an important role in the calculation of generic cohomology. Let

$$H_i=\coprod_{n\in\mathbb{Z}^+} H_i^n, \text{ where } H_i^n=H^n(G_i, K),$$

i. e. H_i is the cohomology ring of G_i. Then H_i is also a rational G–algebra whose G–structure is defined via the isomorphism $G/G_i\cong G$ induced by F^i (this G–structure is usually denoted by $H^{(-i)}$, see [3], for example).

Proposition 3.5. $H\cong\varinjlim H_i$ *as rational* G*–algebras, where the homomorphism*

$H_i \to H_j$ for $i \leq j$ is induced by $\varkappa^{ji} = F^{j-i}|_{G_j}: G_j \to G_i$.

Proof Apply (2.5) to the inverse system $(G_i, \varkappa^{ji})$ and the constant direct system of G_i-modules (K, id). It is not difficult to see that the G-actions are compatible with the procedure of taking direct limit.

Now we see that the G-algebra H defined here is just that defined by Friedlander and Parshall [3] when G is reductive. The following is a generalization of [3, (1.5)] (but we do not know whether H^n is finite dimensional or not, so we cannot state the result in terms of Hom functors as Friedlander and Parshall did in [3]):

Proposition 3.6. *If I is a rationally injective G-module, then*

$$H^n_{\text{gen}}(G, I) \cong (H^n \otimes I)^G.$$

Proof Use spectral sequence (3.4) which collapses when I is rationally injective.

* * *

S. Donkin [2] pointed out that [5, Th. 3.1] means

$$H^n_{\text{gen}}(G, V) \cong H^n(\widetilde{G}, V),$$

where $\widetilde{G}$ is defined by him as the affine group scheme having affine algebra A which we defined in Section 3. He also pointed out that the kernel of $\widetilde{G} \to G$ will play a role in the theory of generic cohomology because of the spectral sequence (3.3). I would like to express my thanks to him for his letter.

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