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# THE GLOBAL SOLUTION OF INITIAL-BOUNDARY VALUE PROBLEM OF HIGHER-ORDER MULTIDIMENSIONAL EQUATION OF CHANGING TYPE

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## Abstract

In practical problems there appears the higher-order equations of changing type. But, there is only a few of papers, which studied the problems for this kind of equations.

In this paper a kind of the higher-order multidimensional equations of changing type is considered:

$$Lu\equiv k(t)u_{tt}+(-1)^{M-1}\sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}(x)D_x^\beta u)-b(x,t)u_t=f(x,t),$$

where $M\geqslant 1$, $x=(x_1,\cdots,x_n)\in\Omega\subset\mathbb{R}^n$, $t\in[0,T]$, $k(t)>0$ when $t=0$, $k(t)\geqslant 0$ when $t\in(0,t_0)$, $k(t)=0$ when $t=t_0$ and $k(t)\leqslant 0$ when $t\in(t_0,T]$.

The existence and uniqueness of the global regular solution of the initial-boundary value problem

$$\begin{cases} D^\nu u=0,\ 0\leqslant|\nu|\leqslant M-1 \text{ on } \partial\Omega\times[0,T],\\ u(x,0)=0,\ x\in\Omega\end{cases}$$

for this equation are proved. Moreover, the result is generalized to a semi-linear equation.

In practical problems there appear the higher-order equations of changing type. For example, they appear in the problems of the flow of viscoelastic fluids (see [1] and the references in it). But, there is only a few of papers, which studied the boundary value problems for the higher-order equations of changing type ([2—4]). In [5] we studied the existence and uniqueness of the global classical solution and $C^\infty$ solution for the initial-boundary value problem of a class of higher-order equations of changing type in two variables. In this paper we study the higher-order multidimensional equation of changing type and generalize it to a semi-linear equation.

In cylindrical domain $Q_T=\{(x,t)\,|\,x\in\Omega,\ 0\leqslant t\leqslant T\}$ we consider the equation

$$Lu\equiv k(t)u_{tt}+(-1)^{M-1}\sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}(x)D_x^\beta u)-b(x,t)u_t=f(x,t), \tag{1}$$

where $M\geqslant 1$, $x=(x_1,\cdots,x_n)\in\Omega\subset\mathbb{R}^n$, $t\in[0,T]$, $D_x^\alpha=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$, $\alpha=(\alpha_1,\cdots,\alpha_n)$,

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$|\alpha|=\alpha_1+\cdots+\alpha_n$, $\alpha_i\geqslant 0$. We study the following initial-boundary value problem

$$\begin{cases} D^\nu u=0,\ 0\leqslant|\nu|\leqslant M-1 \text{ on } \partial\Omega\times[0,T], \\ u(x,0)=0,\ x\in\Omega. \end{cases} \tag{2}$$

Assume that the coefficients and functions in equation (1) satisfy the following conditions:

$$\begin{cases} \text{(i)}\ k(t)>0 \text{ when } t=0,\ k(t)\geqslant 0 \text{ when } t\in(0,t_0),\ k(t)=0 \text{ when} \\ \qquad t=t_0,\ k(t)\leqslant 0 \text{ when } t\in(t_0,T].\ k\in C^1([0,T]). \\ \text{(ii)}\ \left(\sum_{|\alpha|,|\beta|=M} a_{\alpha\beta}(x)D_x^\alpha u,\ D_x^\beta u\right)\geqslant\sigma|D_x^M u|^2_{L_2(\Omega)},\ \sigma>0, \\ \qquad |D_x^M u|_2^2=\sum_{|\alpha|=M}|D_x^\alpha u|_2^2,\ a_{\alpha\beta}(x)=a_{\beta\alpha}(x)\in C^M(\Omega). \\ \text{(iii)}\ b>0,\ b-\frac{1}{2}|k'|\geqslant b_0>0,\ \forall(x,t)\in Q_T, \\ \qquad b\in L_\infty(C^M(\overline{\Omega})\times[0,T]),\ f\in L_\infty(L_2(\overline{\Omega})\times W_2^1([0,T])). \end{cases} \tag{3}$$

We define

$$|u|^2_{L_2(\Omega)}=(u,u)(t)=\int_\Omega u^2dx,\quad \|u\|^2_{L_2(Q_t)}=[u,u](t)=\int_0^t(u,u)dt. \tag{4}$$

The types of equation (1) in different domains, $Q_{t_0}$ and $Q_T\backslash Q_{t_0}$, are different. For example, in the case $M=1$, (1) is of elliptic type in the $Q_{t_0}$, and is of hyperbolic type in the $Q_T\backslash Q_{t_0}$, $t=t_0$ is its degenerate line. In general case ($M>1$), (1) is of hypoelliptic type[6] in domain $Q_{t_0}$, and is of ultra-hyperbolic type[7] in domain $Q_T\backslash Q_{t_0}$.

Assume that on the degenerate line $t=t_0$ the following normal connected conditions are satisfied:

$$\begin{cases} \lim\limits_{t\to t_0-0} -(k(t)u_t,\ u_t)=\lim\limits_{t\to t_0+0}(-k(t)u_t,\ u_t), \\ \lim\limits_{t\to t_0-0} -(k(t)D_x^M u_t,\ D_x^M u_t)=\lim\limits_{t\to t_0+0}(-k(t)D_x^M u_t,\ D_x^M u_t), \\ \lim\limits_{t\to t_0-0} -(k(t)u_{tt},\ u_{tt})=\lim\limits_{t\to t_0+0}(-k(t)u_{tt},\ u_{tt}). \end{cases} \tag{5}$$

**Lemma 1.** *Under conditions (3) and (5), any solution of problem (1) (2) satisfies the uniform estimates:*

$$\begin{cases} \text{(i)}\ (-k(t)u_t,\ u_t)(t)\leqslant\text{const.},\ \forall t\geqslant t_0, \\ \text{(ii)}\ (D_x^M u,\ D_x^M u)(t)\leqslant\text{const.},\ \forall t\geqslant t_0, \\ \text{(iii)}\ [u_t,\ u_t](t)\leqslant\text{const},\ \forall t\in[0,T], \\ \text{(iv)}\ (u_t(x,0),\ u_t(x,0))\leqslant\text{const.}, \end{cases} \tag{6}$$

*where the constants depend on $b_0$ and $[f,f]$ only.*

*Proof* Making integration $[Lu,\ -u_t](t)$, $\forall t\geqslant t_0$ we have

$$-\frac{1}{2}(k(t)u_t,\ u_t)(t)+\frac{1}{2}(k(t)u_t,\ u_t)(0)+\left[\left(b+\frac{1}{2}k'\right)u_t,\ u_t\right](t)$$
$$+\frac{1}{2}\left(\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}(x)D_x^\alpha u, D_x^\beta u\right)(t)=-[f,\ u_t](t),\ \forall t\geqslant t_0. \tag{7}$$

By assumption (3) (i) $k(t)\leqslant 0$ when $t\geqslant t_0$, $k(0)>0$, hence the first two terms in (7) are nonnegative. In consideration of the estimate

$$|[f, u_t](t)| \leqslant \varepsilon[u_t, u_t](t) + C(\varepsilon)[f, f](t), \ \forall t \geqslant t_0,$$

where $\varepsilon = \frac{1}{2} b_0$, by assumption (3) (iii) it follows that the third term in (7) is nonnegative, and moreover, it overcomes the right-hand side. The nonnegativity of the fourth term in (7) is guaranteed by the assumption (3) (ii). Then, follows (6). It is needed to explain that from (7) follows directly $[u_t, u_t](t) \leqslant$ const., $\forall t \geqslant t_0$. In consideration of

$$[v, v](t_1) \leqslant [v, v](t_2), \ \forall t_1 < t_2, \tag{8}$$

we get (6) (iii). Notice that, in the procedure of the integration for $t$ from 0 to $T$, we have to consider the connected conditions (5) on the degenerate line $t = t_0$.

**Lemma 2.** *There is a uniform estimate*

$$[D_x^M u, D_x^M u](t) \leqslant \text{const.} \ \forall t \in [0, T], \tag{9}$$

*where the constant depends on* $b_0$, $\sigma$, $b$, $k$ *and* $k'$ *only.*

*Proof* Making integration $[Lu, -u](t)$, $\forall t \geqslant t_0$ we have

$$-k(t)u_t, u)(t) + (k(t)u_t, u)(0) - [k(t)u_t, u_t](t) + [k'u, u_t](t)$$
$$+ [\sum_{|\alpha|, |\beta| = M} a_{\alpha\beta}(x) D^\alpha u, D^\beta u](t) + [bu_t, u](t) = -[f, u](t), \ \forall t \geqslant t_0. \tag{10}$$

Since

$$|(-k(t)u_t, u)(t)| \leqslant (-k(t)u_t, u_t)(t) + (-k(t)u, u)(t), \ \forall t \geqslant t_0,$$
$$|(k(t)u_t, u)(0)| \leqslant (k(t)u_t, u_t)(0) + (k(t)u, u)(0) = (k(t)u_t, u_t)(0)$$
$$(u, u)(t) \leqslant 2(u, u)(0) + 2t \int_0^t (u_t, u_t) dt \leqslant 2T[u_t, u_t](t), \ \forall t \in [0, T], \tag{11}$$

hence, in consideration of (3) (ii), (6) and (8), from (10) follows (9).

**Lemma 3.** *There hold the following uniform estimates:*

$$\begin{cases} \text{(i)} & (-k(t) D_x^M u_t, D_x^M u_t)(t) \leqslant \text{const.}, \ \forall t \geqslant t_0, \\ \text{(ii)} & (D_x^{2M} u, D_x^{2M} u)(t) \leqslant \text{const.}, \ \forall t \geqslant t_0, \\ \text{(iii)} & [D_x^M u_t, D_x^M u_t](t) \leqslant \text{const.}, \ \forall t \in [0, T], \\ \text{(iv)} & (D_x^M u_t, D_x^M u_t, )(0) \leqslant \text{const.}, \\ \text{(v)} & [D_x^{2M} u, D_x^{2M} u](t) \leqslant \text{const.}, \ \forall t \in [0, T]. \end{cases} \tag{12}$$

*Proof* First, make the integration

$$[Lu, (-1)^{M-1} \sum_{|\alpha|, |\beta| = M} D_x^\alpha (a_{\alpha\beta}(x) D_x^\beta u_t)](t) \equiv \frac{1}{2}(-k(t) \sum_{|\alpha|, |\beta| = M} a_{\alpha\beta}(x) D_x^\alpha u_t, D_x^\beta u_t)(t)$$
$$+ \frac{1}{2}(k(t) \sum_{|\alpha|, |\beta| = M} a_{\alpha\beta}(x) D_x^\alpha u_t, D_x^\beta u_t)(0) + \frac{1}{2} \sum_{|\alpha|, |\beta| = M} D_x^\alpha (a_{\alpha\beta}(x) D_x^\beta u), \sum_{|\alpha|, |\beta| = M} D_x^\alpha (a_{\alpha\beta}(x) D_x^\beta u))(t)$$
$$+ \left[\left(b + \frac{1}{2} k'\right) \sum_{|\alpha|, |\beta| = M} a_{\alpha\beta}(x) D_x^\alpha u_t, D_x^\beta u_t\right](t) + \sum_{|\alpha|, |\beta| = M} \sum_{\substack{|\gamma| + |\delta| = |\alpha| \\ |\delta| < M}} a_{\alpha\beta}(x) c_{\gamma\delta} D_x^\gamma b \cdot D_x^\delta u_t, D_x^\beta u_t](t)$$
$$= (-1)^{M-1} [f, \sum_{|\alpha|, |\beta| = M} D_x^\alpha (a_{\alpha\beta}(x) D_x^\beta u_t)](t), \ \forall t \geqslant t_0. \tag{13}$$

The first two terms in (13) are nonnegative because of $k(t) \leqslant 0$ when $t \geqslant t_0$ and $k(0) > 0$. The integral $[D_x^\delta u_t, D_x^\delta u_t](t)$ $(0 < |\delta| < M)$, obtained from the fifth term, may be estimated by the interpolation formula:

$$[D_x^\delta u_t,\ D_x^\delta u_t](t)\leqslant\varepsilon[D_x^M u_t,\ D_x^M u_t](t)+C(\varepsilon)[u_t,\ u_t](t),\ 0<|\delta|<M.$$

Since such integrals are of finite numbers only, the fifth term may be estimated as follows:

$$\Big|\Big[\sum_{|\alpha|,|\beta|=M}\ \sum_{\substack{|\gamma|+|\delta|=|\alpha|\\|\delta|<M}} a_{\alpha\beta}c_{\gamma\delta}D_x^\gamma b\cdot D_x^\delta u_t,\ D_x^\beta u_t\Big](t)\Big|$$
$$\leqslant\varepsilon_1\sigma[D_x^M u_t,\ D_x^M u_t](t)+C(\varepsilon_1)[u_t,\ u_t](t),\ \forall t\geqslant t_0. \tag{14}$$

The right-hand side in (13) is bounded by

$$\Big|(-1)^{M-1}\Big[f,\ \sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u_t)\Big](t)\Big|\leqslant\Big|\Big(f,\ \sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u)\Big)(t)\Big|$$
$$+\Big|\Big[f_t,\ \sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u)\Big](t)\Big|$$
$$\leqslant\varepsilon_2\Big(\sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u),\ \sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u)\Big)(t)+C(\varepsilon_2)(f,f)(t)$$
$$+\varepsilon_3\Big[\sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u),\ \sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u)\Big](t)+C(\varepsilon_3)[f_t,f_t](t),\ t\geqslant t_0. \tag{15}$$

Summarizing the above estimates, in consideration of (3) (iii), from (13) we have

$$\frac{1}{2}\Big(-k(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t\Big)(t)+\frac{1}{2}\Big(k(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t\Big)(0)$$
$$+\Big(\frac{1}{2}-\varepsilon_2\Big)\Big(\sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u),\ \sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u)\Big)(t)$$
$$+(b_0-\varepsilon_1)\sigma[D_x^M u_t,\ D_x^M u_t](t)$$
$$\leqslant\varepsilon_3\Big[\sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u),\ \sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u)\Big](t)+C(\varepsilon_1)[u_t,\ u_t](t)$$
$$+C(\varepsilon_2)(f,f)(t)+C(\varepsilon_2)[f_t,f_t](t),\ \forall t\geqslant t_0. \tag{16}$$

Second, we make the integration $[Lu,\ (-1)^{M-1}\sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\alpha u)](t)$, and we have

$$\Big(-k(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u\Big)(t)+\Big(k(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u\Big)(0)$$
$$+\Big[k(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t\Big](t)+\Big[k'(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u,\ D_x^\beta u_t\Big](t)$$
$$+\Big[\sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u),\ \sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u)\Big](t)+(-1)^M\Big[bu_t,\ \sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u)\Big](t)$$
$$=(-1)^{M-1}\Big[f,\ \sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}D_x^\beta u)\Big](t),\ \forall t\geqslant t_0. \tag{17}$$

Letting $\sup_{0\leqslant t\leqslant T}|k(t)|=k$, $\sup_{x\in\bar\Omega}|a_{\alpha\beta}(x)|=A$, $\forall t\geqslant t_0$ the first four terms in (17) may be estimated separately as follows:

$$\Big|\Big(-k(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u\Big)(t)\Big|\leqslant\varepsilon_4\Big(-k(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t\Big)(t)$$
$$+C(\varepsilon_4)(D_x^M u,\ D_x^M u)(t),\ \forall t\geqslant t_0, \tag{18}$$

$$\Big|\Big(k(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u\Big)(0)\Big|\leqslant\varepsilon_5\Big(k(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t\Big)(0)$$
$$+C(\varepsilon_5)(D_x^M u,\ D_x^M u)(0), \tag{19}$$

$$\Big|\Big[k(t)\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t\Big](t)\Big|\leqslant KA|D_x^M u_t,\ D_x^M u_t](t),\ \forall t\geqslant t_0, \tag{20}$$

$$|[k'(t)\sum_{|\alpha|,|\beta|=M} a_{\alpha\beta}D_x^\alpha u,\ D_x^\beta u_t](t)| \leqslant [D_x^M u_t,\ D_x^M u_t](t)$$
$$+C[D_x^M u,\ D_x^M u](t),\ \forall t\geqslant t_0. \tag{21}$$

The sixth term and the right-hand side in (17) for $t\geqslant t_0$ are estimated separately by

$$|(-1)^M[bu_t,\ \sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_x^\beta u)](t)|$$
$$\leqslant \varepsilon_6[\sum_{|\alpha|,|\beta|=M} D_x^\alpha u(a_{\alpha\beta}D_x^\beta),\ \sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_x^\beta u)](t)+C(\varepsilon_6)[u_t,\ u_t](t), \tag{22}$$
$$|(-1)^{M-1}[f,\ \sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_x^\beta u)](t)|$$
$$\leqslant \varepsilon_7[\sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_x^\beta u),\ \sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_x^\beta u)](t)+C(\varepsilon_7)[f,\ f](t). \tag{23}$$

Summarizing the above estimates, from (17) we have

$$(1-\varepsilon_6-\varepsilon_7)[\sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_x^\beta u),\ \sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_x^\beta u)](t)$$
$$\leqslant \varepsilon_4(-k(t)\sum_{|\alpha|,|\beta|=M} a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t)(t)+\varepsilon_5(k(t)\sum_{|\alpha|,|\beta|=M} a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t)(0)$$
$$+(KA+1)[D_x^M u_t,\ D_x^M u_t](t)+C(\varepsilon_4,\ \varepsilon_5,\ \varepsilon_6,\ \varepsilon_7)\{[D_x^M u,\ D_x^M u](t)$$
$$+[u_t,\ u_t](t)+[f,\ f](t)+(D_x^M u,\ D_x^M u)(t)+(D_x^M u,\ D_x^M u)(0)\},\ \forall t\geqslant t_0. \tag{24}$$

Multiplying (24) by $\varepsilon_9(>0)$ and adding the result to (16), by choosing

$$\begin{cases} \varepsilon_1=\frac{1}{2}b_0,\ \varepsilon_2=\frac{1}{4},\ \varepsilon_6=\varepsilon_7=\frac{1}{4},\ \varepsilon_4=\varepsilon_5=1/(4\varepsilon_9), \\ \varepsilon_9(KA+1)=\frac{1}{4}b_0\sigma,\ \varepsilon_3=\frac{1}{2}\varepsilon_9(1-\varepsilon_6-\varepsilon_7)=\frac{1}{4}\varepsilon_9, \end{cases} \tag{25}$$

we have

$$\frac{1}{4}(-k(t)\sum_{|\alpha|,|\beta|=M} a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t)(t)+\frac{1}{4}(k(t)\sum_{|\alpha|,|\beta|=M} a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t)(0)$$
$$+\frac{1}{4}(\sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_x^\beta u),\ \sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_\beta^x u))(t)+\frac{1}{4}b_0\sigma[D_x^M u_t,\ D_x^M u_t](t)$$
$$+\frac{1}{4}\varepsilon_9[\sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_x^\beta u),\ \sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}D_x^\beta u)](t)$$
$$\leqslant C(b_0,\ \sigma,\ T)\{(f,\ f)(t)+[f_t,\ f_t](t)+[u_t,\ u_t](t)+(D_x^M u,\ D_x^M u)(t)$$
$$+[D_x^M u,\ D_x^M u](t)+(D_x^M u,\ D_x^M u)(0)\},\ \forall t\geqslant t_0. \tag{26}$$

The boundedness of the right-hand side in (26) is derived from the assumptions (3) and the estimates (6) and (9). The terms on the left-hand side in (26) are all nonnegative. Hence, it implies that all terms on the left-hand side in (26) are bounded for $t\geqslant t_0$, and therefore (12) (i), (ii), (iv) follow directly. Then, by use of (8) we derive the estimate (12) (iii).

As for the estimate (12) (v), we use the character of the uniform strong elliptic operator[8] $Mu(x,\ \cdot)\equiv\sum_{|\alpha|,|\beta|=M} D_x^\alpha(a_{\alpha\beta}(x)D_x^\beta u)$ for $x$

$$|u|^2_{W_2^{2M}(\Omega)}\leqslant C(|Mu|^2_{L_2(\Omega)}+|u|^2_{L_2(\Omega)}),\ \forall u(x,\ \cdot)\in \mathring{W}_2^M(\Omega), \tag{27}$$

and the estimate (derived from (26) and (8)) $[Mu,\ Mu](t)\leqslant$ const., $t\in[0,\ T]$. Integrating (27) with respect to $t$ from 0 to $t(t\in[0,\ T])$, in consideration of (11), we derive (12) (v) immediately.

**Lemma 4.** *We have the following uniform estimates:*

$$\begin{cases}(\mathrm{i}) & (-k(t)u_{tt},\ u_{tt})(t)\leqslant\text{const.},\ \forall t\geqslant t_0,\\ (\mathrm{ii}) & (D_x^M u_t,\ D_x^M u_t)(t)\leqslant\text{const.},\ \forall t\geqslant t_0,\\ (\mathrm{iii}) & [u_{tt},\ u_{tt}](t)\leqslant\text{const.},\ \forall t\in[0,\ T],\\ (\mathrm{iv}) & (u_{tt},\ u_{tt})(0)\leqslant\text{const.}\end{cases} \tag{28}$$

*Proof* Differentiating (1) with respect to $t$ we get

$$(Lu)_t\equiv ku_{ttt}+k'u_{tt}+(-1)^{M-1}\sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}(x)D_x^\beta u_t)-bu_{tt}-b_tu_t=f_t. \tag{29}$$

Making integration $[(Lu)_t,\ -u_{tt}](t)$ we obtain

$$-\frac{1}{2}(k(t)u_{tt},\ u_{tt})(t)+\frac{1}{2}(k(t)u_{tt},\ u_{tt})(0)+\left[\left(b-\frac{1}{2}k'\right)u_{tt},\ u_{tt}\right](t)$$
$$+[b_tu_t,\ u_{tt}](t)+\frac{1}{2}\Big(\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t\Big)(t)-\frac{1}{2}\Big(\sum_{|\alpha|,|\beta|=M}a_{\alpha\beta}D_x^\alpha u_t,\ D_x^\beta u_t\Big)(0)$$
$$=-[f_t,\ u_{tt}](t),\ \forall t\geqslant t_0. \tag{30}$$

From the assumption (3) (i) $k(t)\leqslant 0$ when $t\geqslant t_0$ and $k(0)>0$, the first two terms on the left-hand side in (30) are both nonnegative. Then, from the assumption (3) (ii) and the estimates (6) (iii) and (12) (iv) we derive (28) immediately.

**Theorem 1.** *Under the assumptions* (3) *the initial-boundary value problem* (1) (2) *has a unique global regular solution* $u(x,\ t)\in Z\equiv L_2([0,\ T],\ W_2^{2M}(\Omega))\cap W_2^1([0,\ T],\ W_2^M(\Omega))\cap W_2^2([0,\ T],\ L_2(\Omega))$.

*Proof* By Lemmas 3 and 4 we have estimates

$$\|D_x^M u_t\|_{L_2(Q_T)}\leqslant\text{const.},\ \|D_x^{2M}u\|_{L_2(Q_T)}\leqslant\text{const.},\ \|u_{tt}\|_{L_2(Q_T)}\leqslant\text{const.} \tag{31}$$

Having these estimates we prove the global existence of the regular solution of the problem (1) (2) by the ordinary Galerkin's method[9,10]. Since the equation (1) is linear, the uniqueness of solution is derived from the estimate (12) (iii) immediately: Assume that there are two solutions $u_1$ and $u_2$. Putting $w=u_1-u_2$, then, from (12) (iii) we get $\|D_x^M w_t\|_{L_2(Q_T)}=0$. From this we derive directly $\sup_{x\in\Omega}|D_x^{M-1}w|=0$, $\forall t\in[0,\ T]$. Since $M\geqslant 1$, we have $w\equiv 0$.

Now, we generalize the above result to the semilinear equation

$$\tilde{L}u\equiv k(t)u_{tt}+(-1)^{M-1}\sum_{|\alpha|,|\beta|=M}D_x^\alpha(a_{\alpha\beta}(x)D_x^\beta u)-b(x,\ t)u_t$$
$$=\tilde{f}(x,\ t,\ u,\ Du,\ \cdots,\ D^{M-1}u). \tag{32}$$

Assume that the conditions for the functions $k$, $a_{\alpha\beta}$ and $b$ are still satsfied, and the function $\tilde{f}$, besides (3) (iii), satisfies still more the following two conditions:

$$\begin{cases}(\mathrm{i}) & (\tilde{f},\ \tilde{f})(t)\leqslant C_1,\ t\in[0,\ T],\\ (\mathrm{ii}) & [\tilde{f}_t,\ \tilde{f}_t](t)\leqslant C_2\Big\{1+\sum_{|\nu|=0}^{M-1}[D_x^\nu u_t,\ D_x^\nu u_t](t)\Big\}.\end{cases} \tag{33}$$

Obviously, the conditions (33) are satisfied when $\tilde{f}$ takes the weak nonlinear forms $\sin u$, $\cos D_x^\nu u$, etc.

**Theorem 2.** *Under the assumptions* (3) *and* (33) *the initial boundary value problem* (32) (2) *has a global regular solution* $u(x, t) \in Z$.

*Proof* In order to prove this theorem we need only to add the estimate of $[\tilde{f}_t, \tilde{f}_t]$, which will be instead of $[f_t, f_t]$, appearing in (26). Applying the interpolation formula from (33) (ii) we have

$$[\tilde{f}_t, \tilde{f}_t](t) \leqslant C_2 + \tilde{\varepsilon}_2 [D_x^M u_t, D_x^M u_t](t) + C(\tilde{\varepsilon}_2)[u_t, u_t](t). \tag{34}$$

The second term on the right-hand side of (34) is overcomed by the fourth term on the left-hand side of (26). The third term is finite according to Lemma 1.

**Remark.** It is sufficient if the condition (3) (iii) holds only in $\bar{Q}_{t_0}$.

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