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AUTOMORPHISMS OF $SL(2, K)$ OVER SKEW FIELDS

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Abstract

In this paper, the author proves the following resu:lt

Let K be a skew field and Λ be an automorphism of $SL(2, K)$. Then there exists $A\in GL(2, K)$, an automorphism σ or an anti-automorphism τ of K, such that Λ is of the form

$$\Lambda X=AX^{\sigma}A^{-1} \text{ for all } X\in SL(2, K)$$

or

$$\Lambda X=A(X^{\tau_1})^{-1}A^{-1} \text{ for all } X\in SL(2, K),$$

where X^{σ}, X^{τ} are the matrices obtained by applying σ, τ on X respec tively and X' is the transpose of X.

Let K be a skew field, Z be its center, K^* be its group of units and C be the commutator subgroup of K^*. Let $SL(2, K)$ be the subgroup of $GL(2, K)$ consisting of matrices of determinant 1 (mod C). It is well known that $SL(2, K)$ is equal to $E_2(K)$, the subgroup of $GL(2, K)$ generated by elementary matrices.

Let Λ be an automorphism of $SL(2, K)$. Then Λ is called standard if there exist a matrix $A\in GL(2, K)$ and an automorphism σ or an anti-automorphism τ of K, such that

$$\Lambda X=AX^{\sigma}A^{-1} \quad \text{for all } X\in SL(2, K)$$

or

$$\Lambda X=A({}^tX^{\tau})^{-1}A^{-1} \quad \text{for all } X\in SL(2, K),$$

where if we write $X=(x_{ij})$, then X^{σ} (resp. X^{τ}) is the matrix (σx_{ij}) (resp. (τx_{ij})), and ${}^tX$ is the transpose of X.

It is known that all automorphisms of $SL(2, K)$ are standard if one of the following conditions holds:

(i) the characteristic of K is positive;
(ii) K is commutative;
(iii) -1 is in C; } [2]
(iv) K contains $\sqrt{-1}$, (see [3]);

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(v) K contains $\sqrt{-3}$, (see [4]).

The purpose of the present paper is to prove the follwing theorem.

Theorem. *Let K be any skew field. Then all automorphisms of $SL(2, K)$ are standard.*

Before giving the proof of the theorem, we prove several lemmas needed later. In the sequel, we will write

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ S=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ W=\begin{pmatrix}1 & 1/2\\ 0 & 1\end{pmatrix}.$$

Lemma 1. *Let K be a skew field of characteristic 0, Λ be an automorphism of $SL(2, K)$ and Σ be the set $\left\{\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix}\in SL(2, K)\right\}$. If $\Lambda\Sigma=\Sigma$, then Λ is standard.*

Proof Since $\Lambda\Sigma=\Sigma$, Λ leaves invariant the centralizer of Σ and hence leaves invariant the commutator subgroup of the centralizer of Σ. The centralizer of Σ consists of all matrices whose entries are in the center Z of K. Since $SL(2, Z)$ is the commutator subgroup of both $SL(2, Z)$ and $GL(2, Z)$ and since the centralizer of Σ lies between $SL(2, Z)$ and $GL(2, Z)$, the commutator subgroup of the centralizer of Σ is $SL(2, Z)$. Therefore Λ induces an automsorphim of $SL(2, Z)$. By a known result, $\Lambda|SL(2, Z)$ is standard. So after a conjugation we may assume that $\Lambda|SL(2, Z)$ is of the form $\Lambda X=X^{\sigma}$ or $\Lambda X=({}^tX^{\tau})^{-1}$. In particular,

$$\Lambda\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\ \Lambda\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix};$$

or

$$\Lambda\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix},\ \Lambda\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}.$$

Conjugating by S in the latter case, we may assume that

$$\Lambda\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\text{ and }\Lambda\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

Hence $\Lambda S=S$ and $\Lambda\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ is of the form $\begin{pmatrix}u_x & v_x\\ 0 & u_x\end{pmatrix}$ for some u_x, v_x in K. Since

$$\begin{pmatrix}1 & x^2\\ 0 & 1\end{pmatrix}^{-1}=-S\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix}S\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}S\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}^{-1}S\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix}^{-1}S,$$

we can form corresponding equations for u_x, v_x, $u_{x^{-1}}$, $v_{x^{-1}}$, u_{x^2} and v_{x^2} and get

$$(u_{x^{-1}})^{-1}v_{x^{-1}}=v_x^{-1}u_x,\ v_{x^2}=(v_xu_x^{-1})^2,$$

$u_{x^2}=1$ for all x in K^*. Since

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & (1+x/2)^2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & (x/2)^2\\ 0 & 1\end{pmatrix}^{-1},$$

we get $u_x=1$ for all x in K^*. So $\Lambda\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & \sigma x\\ 0 & 1\end{pmatrix}$, where σ is a bijection of K

satisfying $\sigma(x^2)=(\sigma x)^2$, $\sigma(x^{-1})=(\sigma x)^{-1}$. By equations

$$\begin{pmatrix}1 & x+y\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}$$

and

$$S\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}S\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}S\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}S\begin{pmatrix}1 & x^{-1}\\ 0 & 1\end{pmatrix}S\begin{pmatrix}1 & 1-xyx+2x\\ 0 & 1\end{pmatrix}$$
$$\times S\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}S\begin{pmatrix}1 & 1+x^{-1}\\ 0 & 1\end{pmatrix}=-I,$$

we get $\sigma(x+y)=\sigma x+\sigma y$ and $\sigma(xyx)=(\sigma x)(\sigma y)(\sigma x)$. Hence σ is a semiautomorphism. So by a well known theorem, σ is either an automor-phism or an anti-automorphism. In the former case, we have $\Lambda X=X^\sigma$ for all X in $SL(2, K)$. In the latter case, we have $\Lambda X=S({}^tX^\sigma)^{-1}S^{-1}$ for all X in $SL(2, K)$. So Λ is standard. The lemma is proven.

Lemma 2. *Let K be a skew field such that $\sqrt{-1}\notin K$, i. e. $x^2=-1$ has no solution in K. Let $X\in L(2, K)$ be such that $X^2=-I$. Then X is conju-gate to S in $GL(2, K)$ by a matrix of the form* $\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}$.

Proof Write $X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Since $X^2=-I$ and $a^2\neq -1$, we have $c\neq 0$, $a=-c^{-1}dc$ and $b=-c^{-1}(1+d^2)$. So $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}-c^{-1} & -c^{-1}d\\ 0 & 1\end{pmatrix}S\begin{pmatrix}-c & -d\\ 0 & 1\end{pmatrix}$ and the lemma holds.

Lemma 3. *Let K be a skew field such that $\sqrt{-1}\notin K$. Assume that $X\in GL(2, K)$ is conjugate to a matrix of the form $\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}$ in $GL(2, K)$. Then there exists a matrix B in $GL(2, K)$ such that $BSB^{-1}=S$ and BXB^{-1} is of the form* $\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}$.

Proof Assume MXM^{-1} is of the form $\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}$. Let

$$R=\left\{\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\in GL(2, K)\right\},\quad T=\left\{\begin{pmatrix}a & b\\ -b & a\end{pmatrix}\in GL(2, K)\right\}.$$

Then any matrix M in $GL(2, K)$ can be written as AB with $A\in R$, $B\in T$. For by Lemma 2 we have $MSM^{-1}=ASA^{-1}$ for some $A\in R$, and hence $B=A^{-1}M$ commutes with S, so is in T. Now we have $BSB^{-1}=S$ and

$$BXB^{-1}=A^{-1}MXM^{-1}A=A^{-1}\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}A=\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}.$$

The lemma is proven.

Lemma 4. *Let K be a skew field of characteristic 0 such that $\sqrt{-1}\notin K$, $\sqrt{-3}\notin K$, $-1\notin C$. Let Λ be an automorphism of $SL(2, K)$. If $(\Lambda W)^n-I$ is not invertible for some positive integer n, then Λ is standard.*

Proof It is obvious that $\Lambda(-I)=-I$, so by Lemma 2 we may assume that $\Lambda S=S$. Sin ce $(\Lambda W)^n-I$ is not invertible, $(\Lambda W)^n$ is conjugate to a matrix of the form $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and hence so is $(\Lambda W)^{n^2}$. Since W is conjugate to W^{n^2} in $SL(2, K)$, ΛW is conjugate to $(\Lambda W)^{n^2}$, so is also conjugate to a matrix of the form $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$. By Lemma 3 we can conjugate ΛW to a matrix $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, a, $b \in K$, without change of S. Since $(SW^2)^3=I$, $\sqrt{-3}\notin K$, $-1\notin C$, we can check that $a=1, b=1/2$. So Λ fixes S and W. Hence Λ leaves invariant the centralizer $\left\{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in SL(2, K)\right\}$ of $\langle S, W\rangle$. So Λ is standard by Lemma 1. This completes the proof.

Proof of the theorem By the known results, we may make the following assumptions:

(i) the characteristic of K is 0;

(ii) K is not commutative;

(iii) -1 is not in C;

(iv) $\sqrt{-1}$ is not in K;

(v) $\sqrt{-3}$ is not in K.

It is easy to check that

$$S^2=-I,\ (SW^2)^3=I,\ (W^4SWS)^3=I.$$

Let Λ be an automorphism of $SL(2, K)$. Write $S_1=\Lambda S$, $W_1=\Lambda W$. We have

$$S_1^2=-I$$

and $(S_1W_1^2)^3=I$, $(W_1^4T_1W_1S_1)^3=I$. $(S_1W_1^1)^3=I$ means

$$(S_1W_1^2-I)(S_1W_1^2S_1W_1^2+S_1W_1^2+I)=0.$$

If $S_1W_1^2-I$ is not invertible, then we can conjugate $S_1W_1^2$ to a matrix of the form $\begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}$ in $GL(2, K)$. $\begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}^3=I$ and $\sqrt{-3}\notin K$ means $v=1$, $3u=0$. So $S_1W_1^2=I$ which is absurd. Hence $S_1W_1^2-I$ is invertible and we get

$$(S_1W_1^2)^2+S_1W_1^2+I=0. \tag{2}$$

Similarly we have

$$(W_1^4S_1W_1S_1)^2+W_1^4S_1W_1S_1+I=0. \tag{3}$$

After a conjugation by an element in $GL(2, K)$ if necessary, we may assume $S_1=S$ by Lemma 2. Write $W_1=\begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

Case 1. $y=0$. In this case we have by (2)

$$\begin{cases} z^2w^2=1, & (4) \\ -z^2(wx+xz)+z^2=0. & (5) \end{cases}$$

So z commutes with w^2 and w commutes with z^2 by (4) and

$$wx+xz=1 \tag{6}$$

by (5). Furthermore, (3) means

$$\begin{cases}(-(w^2+z^2)x+w^4z)(w^2+z^2)w+(w^2+z^2)wz^4w-(w^2+z^2)w=0, & (7)\\ -z^4x(w^2+z^2)w+z^4wz^4w-z^4w+1=0. & (8)\end{cases}$$

Since $\sqrt{-3}\notin K$, $(z^4w)^2+z^4w+1\neq 0$. So $w^2+z^2\neq 0$ by (8). Since w^2+z^2 commutes with both w and z, we can divide $(w^2+z^2)w$ from (7) to get

$$-(w^2+z^2)x+w^4z+wz^4-1=0. \tag{9}$$

So x commutes with w^2+z^2, and $z^4\times(9)\times w-(8)$ gives

$$z^4w^4zw-1=0.$$

Hence $z^5w^5=1$. Combining (4) we get $zw=(z^2w^2)^{-2}(z^5w^5)=1$, i. e. $z=w^{-1}$. Consequently, we know by (9) that x commutes with w. So combining (6) and (9) we get

$$\begin{aligned}0&=-(w^2+z^2)+(w^5z+wz^4-1)x^{-1}\\ &=-(w^2+w^{-2})+(w^3-1+w^{-3})(w+w^{-1})\\ &=w^{-4}(w^5-1)(w^3-1).\end{aligned}$$

Hence $w^{15}-1=0$ and then $W_1^{15}-I$ is not invertible. So A is standard by Lemma 4.

Case 2. $x=0$. Conjugating by S_1, the case is reduced to Case 1.

Case 3. $xy\neq 0$. By [1] p. 219, Theorem 8.4.2, there exists a skew field L extending K and an element $a\in L$ such that

$$axa+za-aw-y=0.$$

Case 3.1. $a^2+1\neq 0$. Write $A=\begin{pmatrix}1 & a\\ -a & 1\end{pmatrix}$. Then A is invertible. Write

$$S_2=AS_1A^{-1}=S_1 \text{ and } W_2=AW_1A^{-1}=\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}.$$

Then S_2 and W_2 also satisfy equations (1), (2) and (3). By the same method as in Case 1, we can show that $W_2^{15}-I$ is not invertible. So $W_1^{15}-I$ is not invertible either. Hence A is standard by Lemma 4.

Case 3.2. $a^2+1=0$. Write

$$A=\begin{pmatrix}1 & 0\\ -a & 1\end{pmatrix},\quad S_2=AS_1A^{-1}=\begin{pmatrix}a & 1\\ 0 & -a\end{pmatrix},\quad W_2=AW_1A^{-1}=\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}.$$

Rewrite $W_2=\begin{pmatrix}w & x\\ 0 & z\end{pmatrix}$. Then (5) and (6) mean

$$\begin{cases}aw^2aw^2+aw^2+1=0, & (10)\\ (w^4awa)^2+w^4awa+1=0. & (11)\end{cases}$$

(10) gives

$$aw^2=w^{-2}a=1,\ w^2a=aw^{-2}-1. \tag{12}$$

Apply (12) to (11). We get

$$w^6-w^4a-(w^6+w^2)\,awawa+w^4awa+1=0.$$

Multiplying this equation by aw and then by a on the right, we get

$$(w^8+w^4)awa+(1-w^2)aw+(w^7+w^3-w^2)a+w^5+w^4=0, \tag{13}$$
$$(1-w^2)awa-(w^8+w^4)aw+(w^5+w^4)a-w^7+w^3-w^2)=0. \tag{14}$$

$(1-w^2)\times(13)-(w^8+w^4)\times(14)$ gives

$$f_1aw+f_2a+f_3=0,$$

where

$$f_1=w^{16}+2w^{12}+w^8+w^4-2w^2+1,$$
$$f_2=-w^{13}-w^{12}-2w^9-w^8+w^7-w^5+w^4+w^3-w^2,$$
$$f_3=w^{15}+2w^{11}-w^{10}-2w^6+w^5+w^4.$$

Case 3.2.1. $f_1=0$.

Case 3.2.1.1. $f_2\neq 0$. In this case a is a rational function in w. So $aw=wa$. This means by (10) and (11) that $(aw^2)^3=1$, $(a^2w^5)^3=1$. So $a^3=(aw^2)^{15}(a^2w^5)^{-6}=1$. Thus $-1=a^6=1$ which is absurd. So this case will not occur.

Case 3.2.1.2. $f_2=0$. In this case $f_3=0$. So

$$0=w^4f_1-(w^5+1)f_3=w^5(w^4+1)(w^3-1).$$

Hence $w^{24}-1=0$, and then $W^{24}-I$ is not invertible. So $W_1^{24}-I$ is not invertible. Thus Λ is standard by Lemma 4.

Case 3.2.2. $f_1\neq 0$. In this case we have

$$aw=-f_1^{-1}f_2a-f_1^{-1}f_3.$$

So by (12)

$$w^{-2}a-1=aw^2=(-f_1^{-1}f_2a-f_1^{-1}f_3)w$$
$$=-f_1^{-1}f_2(-f_1^{-1}f_2a-f_1^{-1}f_3)-f_1^{-1}f_3w.$$

Hence

$$(f_1^2-w^2f_2^2)a=w^2(f_2f_3-wf_1f_3+f_1^2).$$

Case 3.2.2.1. $f_1^2-w^2f_2^2\neq 0$. In this case a commutes with w. Hence we can get a contradiction as in Case 3.2.1.1.

Case 3.2.2.2. $f_1^2-w^2f_2^2=0$. Then $f_1=\varepsilon wf_2$, $\varepsilon=\pm 1$. We have by (15)

$$0=f_2f_3=wf_1f_3+f_1^2=f_2(f_3-\varepsilon w^2f_3+w^2f_2).$$

$f_1\neq 0$ means $f_2\neq 0$, So $f_3-\varepsilon w^2f_3+w^2f_2=0$. Hence

$$0=w^3(f_1-\varepsilon wf_2)+(\varepsilon w^2+1)(f_3-\varepsilon w^2f_3+w^2f_2)=w^3(w^5-1)(w^3-1).$$

This means that $w^{15}-1=0$ and hence that $w_2^{15}-I$ is not invertible. So $W_1^{15}-I$ is not invertible and hence Λ is standard by Lemma 4. This completes the proof of the theorem.

References

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