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FRAME MAPPINGS AND POINT SEPARATION AXIOMS OF TOPOLOGICAL SPACES**

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Abstract

In this paper, the author uses frame mappings to describe the T_0 T_1 and T_2 axioms, and also gets a characterization of Sober space.

Following Wallman's idea of studying topological spaces and properties with lattice theoretic point of view, C. Ehresmann and his student J. Bénabou first took complete Heyting algebra as generalized topological space to study. After that, C. H. Dowker, D. Papert, J. R. Isbell and many others began a great deal of studies on complete Heyting algebra, and set up a new branch of mathematics-frame theory (or its dual category: locale theory)[2]. With the help of category, frame theory makes algebra topologized, and also turns topology in a more algebraic way. Its theory has close relations with some active branches of mordern mathematics such as category, sheaf and topos theories. And its application can be found in the areas of continuous lattices developed from computer theory, of lattice-topology with the background of fuzzy mathematics[3], and of generalizde topological molecular lattices[5]. The refore it not only draws wide interests abroad, but is also energetically encouraged by Prof. Liu Ying-ming and Wang Guo-jun in our country.

In this paper, we use frame methods to deal with the point separation axioms of topological spaces, that is, we describe the T_0, T_1, and T_2 axioms with frame mappings. And when describing the Soberity, we give out an equivalent condition when a frame mapping can be "lifted" to a continuous mapping.

We call A a frame if A is a complete lattice satisfying the infinite distributive law: for any a, $a_t \in A$, $t \in T$, $a \wedge \bigvee_{t \in T} a_t = \bigvee_{t \in T} a \wedge a_t$. Frm is the category of frames with morphisms of arbitrarily join-and finitely meet-preserving mappings between frames (which are called frame mappings). If a frame mapping is also arbitrarily meet-preserving, we call it a complete mapping.

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Given two topological spaces $(X^*, \mathcal{O}^*)$, $(X, \mathcal{O})$ and a continuous mapping f: $X^* \to X$, then the sets $\mathcal{O}^*$ and $\mathcal{O}$ of open subsets of X^* and X are frames and the mapping f^{-1}: $\mathcal{O} \to \mathcal{O}^*$ is a frame mapping. We call f^{-1}: $\mathcal{O} \to \mathcal{O}^*$ the frame mapping of f: $X^* \to X$. Now we natrurally want to know when the mapping f: $X^* \to X$ can be decided by its frame mpping f^{-1}: $\mathcal{O} \to \mathcal{O}^*$.

Theorem 1. *A topological space $(X, \mathcal{O})$ is a T_0 space iff for any topological space $(X^*, \mathcal{O}^*)$ and mappings f, g: $X^* \to X$, $f = g$: $X^* \to X$ is equivalent to $f^{-1} = g^{-1}$: $\mathcal{O} \to \mathcal{O}^*$.*

Proof If $(X, \mathcal{O})$ is a T_0 space, and mappings f, g: $X^* \to X$ such that $f^{-1} = g^{-1}$: $\mathcal{O} \to \mathcal{O}^*$, and if there exists $x \in X^*$ such that $f(x) \neq g(x)$, then since X is T_0, we have $U \in \mathcal{O}$, U containing only one of $f(x)$ and $g(x)$. Suppose $f(x) \in U$. Then $x \in f^{-1}(U) = g^{-1}(U)$. Thus $g(x)$ is also contained in U. This is a contradiction to the property of U.

Now if $(X, \mathcal{O})$ is not a T_0 space, then there exist different x and y in X such that for any $U \in \mathcal{O}$, $x \in U$ if and only if $y \in U$. Let $X^* = \{\infty\}$, $\mathcal{O}^* = \{\phi, X^*\}$, and let f: $X^* \to X$, $\infty \mapsto x$; g: $X^* \to X$, $\infty \mapsto y$. Clearly $f^{-1} = g^{-1}$: $\mathcal{O} \to \mathcal{O}^*$ but $f \neq g$. This shows that the hypothesis dose not hold.

The proofs of the following two theorems are left to the reader.

Theorem 2. *A topological space $(X, \mathcal{O})$ is T_0 iff there exists a topological space $(X^*, \mathcal{O}^*)$ such that for any mappings f, g: $X^* \to X$, $f = g$ is equivalent to $f^{-1} = g^{-1}$: $\mathcal{O} \to \mathcal{O}^*$.*

Theorem 3. *A space $(X, \mathcal{O})$ is T_0 iff the following condition holds: a mapping f: $X \to X$ is identical iff the mapping f^{-1}: $\mathcal{O} \to \mathcal{O}$ is identical.*

Corollary. 1 *If both $(X, \mathcal{O})$ and $(X^*, \mathcal{O}^*)$ are T_0 spaces, then f: $X \to X^*$ is the homeomorphic inverse mapping of g: $X^* \to X$ iff f^{-1}: $\mathcal{O}^* \to \mathcal{O}$ is the isomorphic inverse mapping of g^{-1}: $\mathcal{O} \to \mathcal{O}^*$.*

Now we know when a mapping is decided by its frame mapping. But we naturally ask when a frame mapping between two frames of open subsets can give out a continuous mapping.

Theorem 4. *A topological space $(X, \mathcal{O})$ is a Sober space iff for any space $(X^*, \mathcal{O}^*)$ and frame mapping k: $\mathcal{O} \to \mathcal{O}^*$, there exists a unique continuous mapping f: $X^* \to X$ such that $f^{-1} = k$: $\mathcal{O} \to \mathcal{O}^*$.*

Proof Suppose that $(X, \mathcal{O})$ is a Sober space, for any frame mapping k: $\mathcal{O} \to \mathcal{O}^*$ and for any $x \in X^*$, let $G_x = \subset\{G \in \mathcal{O}$: $x \notin k(G)\}$. Clearly G_x is the largest open subset such that $x \notin k(G_x)$. It is easy to show that G_x is a prime open subset. Since X is Sober, there exists a unique y_x such that $G_x = X \setminus \overline{\{y_x\}}$. Let $f(x) = y_x$ for each $x \in X^*$. Then we get a mapping f: $X^* \to X$. Now for any $U \in \mathcal{O}$, if $x \notin k(U)$, then $U \subset G_x$, thus $f(x) = y_x \notin U$, hence $x \notin f^{-1}(U)$. Now if $x \notin f^{-1}(U)$, then $f(x) = y_x \notin U$.

So $U\subset X\setminus\overline{\{y_x\}}=G_x$. Since $x\notin k(G_x)$, we have $x\notin k(U)$. Therefore $f^{-1}=k$: $\mathscr{O}\to\mathscr{O}^*$. By this we also know that f: $X^*\to X$ is continuous. Since a Sober space is T_0, the uniqueness is given by Theorem 1.

Now suppose $(X, \mathscr{O})$ satisfies the hypothesis. We know it is T_0 by Theorem 1. To show $(X, \mathscr{O})$ is Sober we need only to show the mapping j: $X\to\mathrm{Pr}(\mathscr{O})$, $x\mapsto X\setminus\overline{\{x\}}$ is surjective, where $\mathrm{Pr}(\mathscr{O})$ is the set of all pri me open subsets V of X, i. e. $V\supset V_1\cap V_2$ implies $V\supset V_1$ or $V\supset V_2$ for all $V_1, V_2\in\mathscr{O}$. Now it is easy to show h: $\mathscr{O}\to P(\mathrm{Pr}(\mathscr{O}))$, $U\mapsto\{V\in\mathrm{Pr}(\mathscr{O})\colon V\not\supset U\}$ is a frame mapping. Thus the image $h(\mathscr{O})$ is a topology on the set $\mathrm{Pr}(\mathscr{O})$. By the hypothesis we have f: $(\mathrm{Pr}(\mathscr{O}), h(\mathscr{O}))\to(X, \mathscr{O})$ such that $f^{-1}=h$: $\mathscr{O}\to h(\mathscr{O})$. Since for any $W\in h(\mathscr{O})$ there exists $U\in\mathscr{O}$ such that $h(U)=W$, we have

$$f^{-1}\circ j^{-1}(W)=h(j^{-1}(h(U)))=h(\{x\in X\colon j(x)\in h(U)\})$$
$$=h(\{x\in X;\ X\setminus\overline{\{x\}}\not\supset U\})=h(U)=W.$$

Thus $(j\circ f)^{-1}$: $h(\mathscr{O})\to h(\mathscr{O})$ is identical. Since $(\mathrm{Pr}(\mathscr{O}), h(\mathscr{O}))$ is a T_0 space, by Theorem 3 we know $j\circ f$: $\mathrm{Pr}(\mathscr{O})\to\mathrm{Pr}(\mathscr{O})$ is also identical. Therefore j is surjective.

Corollary 2. *If $(X^*, \mathscr{O}^*)$ is Sober and $(X, \mathscr{O})$ T_0, then f: $X^*\to X$ is a homeomorphism iff f^{-1}: $\mathscr{O}\to\mathscr{O}^*$ is an isomorphism.*

This can be proved by the theorem above and Corollary 1.

We find that for any open mapping f: $X^*\to X$, its frame mapping f^{-1}: $\mathscr{O}\to\mathscr{O}^*$ is complete, i.e. for an y $U_t\in\mathscr{O}$, $t\in T$ $f^{-1}(\mathrm{Int}_{\mathscr{O}}\bigcap_{t\in T}U_t)=\mathrm{Int}_{\mathscr{O}^*}\bigcap_{t\in T}f^{-1}(U_t)$. Then when is the inverse true?

Theorem 5. *A topological space $(X, \mathscr{O})$ is a T_1 space iff the following condition holds: for any space $(X^*, \mathscr{O}^*)$ and f: $X^*\to X$, f is open iff f^{-1}: $\mathscr{O}\to\mathscr{O}^*$ is complete.*

Proof If $(X, \mathscr{O})$ is a T_1 space and f^{-1}: $\mathscr{O}\to\mathscr{O}^*$ complete, then for any subset A o f X, by T_1 axiom $A=\cap\{U\in\mathscr{O}\colon U\supset A\}$. Thus

$$f^{-1}(\mathrm{Int}_{\mathscr{O}}A)=f^{-1}(\mathrm{Int}_{\mathscr{O}}\cap\{U\colon U\supset A\}$$
$$=\mathrm{Int}_{\mathscr{O}^*}\cap\{f^{-1}(U)\colon U\supset A\}$$
$$=\mathrm{Int}_{\mathscr{O}^*}f^{-1}(\cap\{U\colon V\supset A\}$$
$$=\mathrm{Int}_{\mathscr{O}^*}f^{-1}(A).$$

Hence f: $X^*\to X$ is an open mapping.

Now if $(X, \mathscr{O})$ is not a T_1 space, then there exists a point $x^*\in X$ such that $\{x^*\}$ is not a closed subset. Let $X^*=X\setminus\{x^*\}$ and $\mathscr{O}^*$ is the topology of the subspace X^* of X. Then the inclusion mapping i: $X^*\to X$ is continuous but not open, while for any $U_t\in\mathscr{O}$, $t\in T$, we have

$$\mathrm{Int}_{\mathscr{O}^*}\bigcap_{t\in T}i^{-1}(U_t)=\mathrm{Int}_{\mathscr{O}^*}\bigcap_{t\in T}(U_t\cap X^*)$$
$$=\mathrm{Int}_{\mathscr{O}^*}\bigcap_{t\in T}U_t\cap X^*=X^*\cap\mathrm{Int}_{\mathscr{O}}\bigcap_{t\in T}U_t=i^{-1}(\mathrm{Int}_{\mathscr{O}}\bigcap_{t\in T}U_t)$$

So i^{-1}: $\mathscr{O}\to\mathscr{O}^*$ is complete. This shows that the sufficiency is true.

Corollary 3. *If $(X, \mathscr{O})$ is T_1 and Sober, particularly if it is T_2, then for any space $(X^*, \mathscr{O}^*)$ add complete mapping k: $\mathscr{O}\to\mathscr{O}^*$, k decides an open mapping f: $X^*\to X$ such that $f=\hat{k}$: $\mathscr{O}^*\to\mathscr{O}$,* where $\hat{k}$: $\mathscr{O}^*\to\mathscr{O}$ $U\mapsto \mathrm{Int}_{\mathscr{O}}\cap\{V:K(V)\supset U\}$.

Proof This is by Theorem 4, Theorem 5, and [4].

Now we give out a description of T_2 axiom which also has similar conclusions for T_0 and T_1 axioms.

Definition. *A frame mapping k: $A\to B$ is said to be T_2 if for any different prime elements p, $q\in \mathrm{Pr}(B)$, there exist a, $b\in A$ such that $p\not\geqslant k(a)$ and $q\not\geqslant k(b)$ while there is no r in $\mathrm{Pr}(B)$ satisfying $r\not\geqslant k(a)$ and $r\not\geqslant k(b)$.*

Proposition. *If $(X, \mathscr{O})$ is a Sober space, for any space $(X^*, \mathscr{O}^*)$ and mapping f: $X\to X^*$, f^{-1}: $\mathscr{O}^*\to\mathscr{O}$ is a T_2 frame mapping iff for any different x, $y\in X$ there exist U, $V\in\mathscr{O}^*$ such that $x\in f^{-1}(U)$, $y\in f^{-1}(V)$ and $f^{-1}(U)\cap f^{-1}(V)=\varnothing$.*

Proof If f^{-1}: $\mathscr{O}^*\to\mathscr{O}$ is a T_2 mapping, then for any different x, $y\in X$, since X is T_0, we know $X\setminus\overline{\{x\}}$ and $X\setminus\overline{\{y\}}$ are different elements in $\mathrm{Pr}(\mathscr{O})$. By definition there exist U, $V\in\mathscr{O}^*$ such that $X\setminus\overline{\{x\}}\not\supset f^{-1}(U)$, $X\setminus\overline{\{y\}}\not\supset f^{-1}(V)$ and there is no $W\in\mathrm{Pr}(\mathscr{O})$ satisfying $W\not\supset f_-^1(U)$ and $W\not\supset f^{-1}(V)$. Thus $x\in f^{-1}(U)$ and $y\in f^{-1}(V)$. Now if $z\in f^{-1}(U)\cap f^{-1}(V)$, then $X\setminus\overline{\{z\}}\in\mathrm{Pr}(\mathscr{O})$ and $X\setminus\overline{\{z\}}\not\supset f^{-1}(U)$, $X\setminus\overline{\{z\}}\not\supset f^{-1}(V)$. This is known to be impossible. Thus $f^{-1}(U)\cap f^{-1}(V)=\varnothing$.

Conversely, if f: $X\to X^*$ satisfies the hypothesis, now for any different p, $q\in \mathrm{Pr}(\mathscr{O})$, since X is Sober, there exist x, $y\in X$ such that $p=X\setminus\overline{\{x\}}$ and $q=X\setminus\overline{\{y\}}$. By hypothesis, there exist U, $V\in\mathscr{O}^*$ such that $x\in f^{-1}(U)$, $y\in f^{-1}(V)$ and $f^{-1}(U)\cap f^{-1}(V)=\varnothing$. Thus $p\not\supset f^{-1}(U)$, $q\not\supset f^{-1}(V)$. It is easy to see there is no $r\in\mathrm{Pr}(\mathscr{O})$ satisfying $r\not\supset f^{-1}(U)$ and $r\not\supset f^{-1}(V)$. So f^{-1}: $\mathscr{O}^*\to\mathscr{O}$ is T_2.

Theorem 6. *$(X, \mathscr{O})$ is a T_2 space iff the following condition hol ds*: for any *Sober space $(X^*, \mathscr{O}^*)$ and mapping f: $X^*\to X$, f is monomorphism iff f^{-1}: $\mathscr{O}\to\mathscr{O}^*$ is T_2.*

Proof Necessity. If f^{-1}: $\mathscr{O}\to\mathscr{O}^*$ is T_2, f: $X^*\to X$ is obviously a monomorphism. Now if f: $X^*\to X$ is a monomorphism, then for any different x, $y\in X^*$, $f(x)\neq f(y)$. Since X is T_2, we have U, $V\in\mathscr{O}$ such that $f(x)\in U$, $f(y)\in V$ and $U\cap V=\varnothing$. Thus $x\in f^{-1}(U)$, $y\in f^{-1}(V)$ and $f^{-1}(U)\cap f^{-1}(V)=\varnothing$. By the proposition above, we know f^{-1}: $\mathscr{O}\to\mathscr{O}^*$ is a T_2 mapping.

Suppose $(X, \mathscr{O})$ is not a T_2 space, we have x, $y\in X$ such that for any U, $V\in\mathscr{O}$, if $x\in U$, $y\in V$, then there exists a point $z(U, V)\in U\cap V$. Let $X^*=\{x, y\}\cup\{z(U, V)$: $(x, y)\in U\times V\in\mathscr{O}\times\mathscr{O}\}$ with the discrete topology $\mathscr{O}^*$. Then $(X^*, \mathscr{O}^*)$ is a Sober space and the inclusion mapping i: $X^*\to X$ is a continuous monomorphism. But for any U, $V\in\mathscr{O}$ such that $x\in i^{-1}(U)$, $y\in i^{-1}(V)$, we have $r=X^*\setminus\{z(U, V)\}\in\mathrm{Pr}(\mathscr{O}^*)$ satisfying $r\not\supset U$ and $r\not\supset V$. This shows that the hypothesis does not hold.

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