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A SUFFICIENT CONDITION FOR ONE CONVEX BODY CONTAINING ANOTHER

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Abstract

A sufficient condition for one convex body to be able to contain another in 3-dimensional Euclidean space has been derived by estimating the kinematic measure of one convex body moving in another. The equality of estimating inequality holds if and only if two convex bodies are congruent balls.

§1. Introduction and Results

In this paper, we have derived a sufficient condition for one convex body to contain another in three-dimensional Euclidean space by estimating the kinematic measure of one convex body moving in another. In the plane, the sufficient condition for one simply connected domain to contain another obtained by Hadwiger[4],[5] is very interesting since it only involves the areas and the perimeters of two simply connected domains. However, there is no similar result about convex bodies in space before our work appears.

Let K_i ($i=0, 1$) be convex bodies in 3-dimensional Euclidean space E^3, ∂K_i its boundary. Assume the boundaries of these convex bodies are class C^2. We denote by V_i the volume of K_i and by F_i, M_i, N_i the area, the integral of mean curvature and the integral of square of mean curvature of ∂K_i respectively. We get the following result:

A sufficient condition for $K_0 \subset K_1$ or $K_1 \subset K_0$ is

$$8\pi^2(V_0+V_1)-\frac{2}{3}\pi(F_0M_1+F_1M_0)-\frac{2}{3}\{2F_0F_1[3(N_0F_1+N_1F_0)-4\pi(F_0+F_1)-4M_0M_1]\}^{1/2}>0. \tag{*}$$

Above result follows an estimation of the kinematic measure m of one convex body moving in another. Namely,

$$m_0 \geqslant 8\pi^2(V_0+V_1)-\frac{2}{3}\pi(F_0M_1+F_1M_0)-\frac{2}{3}\{2F_0F_1[3(N_0F_1+N_1F_0)-4\pi(F_0+F_1)-4M_0M_1]\}^{1/2}, \tag{**}$$

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the equality holds if and only if two convex bodies are congruent balls.

§2. Proof

The proof of (**) is divided into three steps. First, we derive a density formula; next, we estimate the integral of total curvature of the intersection of two convex bodies; we finally complete the proof by applying Fenchel's theorem, Blaschke's formula and the estimation in the second step.

1. Let K_0 be fixed and K_1 move such that $\partial K_0 \cap \partial K_1 \neq \varnothing$. Then $C = \partial K_0 \cap \partial K_1$ is a curve which possiblely consists of several components. Let ds be the length element of C at a point $P \in C$ and ds_0, which perpendicular to C, be the length element of ∂K_0 at P. Let θ be the angle between normal lines of ∂K_0, ∂K_1 and θ_0, θ_1 be the angle between the tangent vector of C and one of the principal directions of ∂K_0, ∂K_1 respectively. Denote by da_0, da_1 the volume element of ∂K_0, ∂K_1 respectively.

If $\{P;\ e_1,\ e_2,\ e_3\}$ be a moving orthonomal frame fixed on K_1, where e_3 is the normal vector of K_1, then the kinematic density of K_1 may be written as ([1], p. 198.)

$$dK_1 = dP \wedge du_2 \wedge du_1, \tag{1}$$

where dP is the volume element of E^3, du_2 is the volume element of the 2-dimensional unit sphere at the end point of e_1 and du_1 is the volume element of the 1-dimensional unit sphere at the end point of e_2.

Because the volume element dP of space E^3 is equal to the exterior product of the surface area element da_1 of ∂K_1 with the length element $\sin\theta ds_0$ of the normal line of ∂K_1, we have

$$dP = \sin\theta\, ds_0 \wedge da_1$$

and hence

$$ds \wedge dP = \sin\theta\, ds \wedge ds_0 \wedge da_1 = \sin\theta\, da_0 \wedge da_1. \tag{2}$$

On surface ∂K_0, consider a fixed orthonormal frame at point P, which is determined by the unit normal vector and the principal directions. About this frame, the unit normal vector e_3 of ∂K_1 may be written as

$$(\sin\theta\cos\theta_0,\ \sin\theta\sin\theta_0,\ \cos\theta).$$

Then the volume element of the 2-dimensional unit sphere at the end point of e_3 is

$$du_2 = \sin\theta\, d\theta \wedge d\theta_0, \quad 0 \leqslant \theta_0 \leqslant 2\pi,\ 0 \leqslant \theta \leqslant \pi. \tag{3}$$

Obviously

$$du_1 = d\theta_1, \quad 0 \leqslant \theta_1 \leqslant 2\pi. \tag{3$'$}$$

From (1) – (3)′, we obtain

$$ds \wedge dK_1 = \sin^2\theta\, da_0 \wedge da_1 \wedge d\theta_0 \wedge d\theta_1 \wedge d\theta. \tag{4}$$

2. Let $\varkappa$ be the curvature of C. We shall estimate the following integral

$$I=\int_{\partial K_0\cap\partial K_1\neq\phi}\left(\int_C \varkappa\, ds\right)dK_1. \tag{5}$$

Throughout, we shall make use of the following convention on the angles:

$$0\leqslant\theta_0\leqslant 2\pi.\ 0\leqslant\theta_1\leqslant 2\pi,\ 0\leqslant\theta\leqslant\pi.$$

We need an expression of $\varkappa$ which involves the normal curvature of ∂K_i, namely ([3], p. 102)

$$\varkappa^2\sin^2\theta=(\varkappa_n^{(0)})^2+(\varkappa_n^{(1)})^2-2\varkappa_n^{(0)}\varkappa_n^{(1)}\cos\theta,$$

where $\varkappa_n^{(i)}$ is the normal curvature of ∂K_i. From this, we have

$$\begin{aligned}\int_0^\pi \varkappa\sin^2\theta\, d\theta&=\int_0^\pi [(\varkappa_n^{(0)})^2+(\varkappa_n^{(1)})^2-2\varkappa_n^{(0)}\varkappa_n^{(1)}\cos\theta]^{1/2}\sin\theta\, d\theta\\&=(3\varkappa_n^{(0)}\varkappa_n^{(1)})^{-1}(|\varkappa_n^{(0)}+\varkappa_n^{(1)}|^3-|\varkappa_n^{(0)}-\varkappa_n^{(1)}|^3).\end{aligned} \tag{6}$$

If $\varkappa_n^{(0)}\geqslant\varkappa_n^{(1)}$, then

$$\begin{aligned}(3\varkappa_n^{(0)}\varkappa_n^{(1)})^{-1}(|\varkappa_n^{(0)}+\varkappa_n^{(1)}|^3-|\varkappa_n^{(0)}-\varkappa_n^{(1)}|^3)&=(3\varkappa_n^{(0)})^{-1}(6(\varkappa_n^{(0)})^2+2(\varkappa_n^{(1)})^2)\\&\leqslant 2(\varkappa_n^{(0)}+\varkappa_n^{(1)})-\frac{4}{3}\varkappa_n^{(1)}.\end{aligned} \tag{7}$$

Similarly, if $\varkappa_n^{(1)}\geqslant\varkappa_n^{(0)}$, we have

$$(3\varkappa_n^{(0)}\varkappa_n^{(1)})^{-1}(|\varkappa_n^{(0)}+\varkappa_n^{(1)}|^3-|\varkappa_n^{(0)}-\varkappa_n^{(1)}|^3)\leqslant 2(\varkappa_n^{(0)}+\varkappa_n^{(1)})-\frac{4}{3}\varkappa_n^{(0)}. \tag{8}$$

Note that the equalities hold if and only if $\varkappa_n^{(0)}=\varkappa_n^{(1)}$. From (6) – (8), we then get

$$\int_0^\pi \varkappa\sin^2\theta\, d\theta\leqslant 2(\varkappa_n^{(0)}+\varkappa_n^{(1)})-\frac{4}{3}\min\{\varkappa_n^{(0)},\ \varkappa_n^{(1)}\}. \tag{9}$$

By (9) and

$$\min\{\varkappa_n^{(0)},\ \varkappa_n^{(1)}\}=\frac{1}{2}(|\varkappa_n^{(0)}+\varkappa_n^{(1)}|-|\varkappa_n^{(0)}-\varkappa_n^{(1)}|)$$

we obtain

$$\int_0^\pi \varkappa\sin^2\theta\, d\theta\leqslant\frac{4}{3}(\varkappa_n^{(0)}+\varkappa_n^{(1)})+\frac{2}{3}|\varkappa_n^{(0)}-\varkappa_n^{(1)}|, \tag{10}$$

the equality holds if and only if $\varkappa_n^{(0)}=\varkappa_n^{(1)}$.

We need Euler's formula

$$\varkappa_n^{(i)}=\varkappa_1^{(i)}\cos^2\theta_i+\varkappa_2^{(i)}\sin^2\theta_i,\quad i=0,\ 1, \tag{11}$$

where $\varkappa_1^{(i)}$, $\varkappa_2^{(i)}$ are the principal curvatures of ∂K_i.

From (4), (5) and (10), we have

$$I\leqslant\int\left[\frac{4}{3}(\varkappa_n^{(0)}+\varkappa_n^{(1)})+\frac{2}{3}|\varkappa_n^{(0)}-\varkappa_n^{(1)}|\right]da_0\wedge da_1\wedge d\theta_0\wedge d\theta_1. \tag{12}$$

By (11) and $\int_0^{2\pi}\cos^2\varphi\, d\varphi=\int_0^{2\pi}\sin^2\varphi\, d\varphi=\pi$, we get

$$\int\frac{4}{3}(\varkappa_n^{(0)}+\varkappa_n^{(1)})\,da_0\wedge da_1\wedge d\theta_0\wedge d\theta_1=\frac{16\pi^2}{3}(M_0F_1+M_1F_0), \tag{13}$$

where

$$M_i=\int_{\partial K_i}\frac{1}{2}(\varkappa_1^{(i)}+\varkappa_2^{(i)})\,da_i,\quad i=0,\ 1.$$

According to Schwarz inequality, we have

$$\int|\varkappa_n^{(0)}-\varkappa_n^{(1)}|\,da_0\wedge da_1\wedge d\theta_0\wedge d\theta_1\leqslant(4\pi^2F_0F_1)^{1/2}\Big(\int(\varkappa_n^{(0)}-\varkappa_n^{(1)})^2\,da_0\wedge da_1\wedge d\theta_0\wedge d\theta_1\Big)^{1/2}. \tag{14}$$

From (11) and

$$\int_0^{2\pi}\cos^4\theta_i\,d\theta_i=\int_0^{2\pi}\sin^4\theta_i\,d\theta_i=\frac{3}{4}\pi,\quad \int_0^{2\pi}\cos^2\theta_i\sin^2\theta_i\,d\theta_i=\frac{\pi}{4}$$

we have

$$\int\varkappa_n^{(i)}\,da_i\wedge d\theta_i=\pi\int(\varkappa_1^{(i)}+\varkappa_2^{(i)})\,da_i=2\pi M_i, \tag{15}$$

$$\int(\varkappa_n^{(i)})^2\,da_i\wedge d\theta_i=3\pi\int\Big(\frac{\varkappa_1^{(i)}+\varkappa_2^{(i)}}{2}\Big)^2da_i-\pi\int\varkappa_1^{(i)}\varkappa_2^{(i)}\,da_i=3\pi N_i-4\pi^2, \tag{16}$$

where $N_i=\int_{\partial K_i}\Big(\frac{\varkappa_1^{(i)}+\varkappa_2^{(i)}}{2}\Big)^2da_i$ and $\int_{\partial K_i}\varkappa_1^{(i)}\varkappa_2^{(i)}\,da_i=4\pi$ is the Gauss–Bonnet formula.

Using (15) and (16), we get

$$\int(\varkappa_n^{(0)}-\varkappa_n^{(1)})^2\,da_0\wedge da_1\wedge d\theta_0\wedge d\theta_1$$
$$=6\pi^2(N_0F_1+N_1F_0)-8\pi^3(F_0+F_1)-8\pi^2M_0M_1. \tag{17}$$

According to (14) and (17), we obtain

$$\int|\varkappa_n^{(0)}-\varkappa_n^{(1)}|\,da_0\wedge da_1\wedge d\theta_0\wedge d\theta_1$$
$$\leqslant 2\pi^2\{F_0F_1[6(N_0F_1+N_1F_0)-8\pi(F_0+F_1)-8M_0M_1]\}^{1/2}. \tag{18}$$

Together with (12), (13), we finally obtain

$$I\leqslant\frac{16}{3}\pi^2(F_0M_1+F_1M_0)$$
$$+\frac{4}{3}\pi^2\{2F_0F_1[3(N_0F_1+N_1F_0)-4\pi(F_0+F_1)-4M_0M_1]\}^{1/2}. \tag{19}$$

The equality in (19) holds if and only if the normal curvature of ∂K_1 at any point and along any direction is equal to that of ∂K_0 and hence ∂K_0 and ∂K_1 must be congruent spheres.

3. According to Fenchel's Theorem

$$\int_C\varkappa\,ds\geqslant 2\pi$$

we have

$$I\geqslant 2\pi\int_{\partial K_0\cap\partial K_1\neq\phi}dK_1. \tag{20}$$

Applying (20) and Blaschke's formula

$$\int_{K_0\cap K_1\neq\phi}dK_1=8\pi^2(V_0+V_1)+2\pi(F_0M_1+F_1M_0)$$

we have

$$m_0=m\{K_1\subset K_0 \text{ or } K_0\subset K_1\}=\int_{K_0\cap K_1\neq\phi}dK_1-\int_{\partial K_0\cap\partial K_1\neq\phi}dK_1$$

$$\geqslant 8\pi^2(V_0+V_1)+2\pi(F_0M_1+F_1M_0)-\frac{1}{2\pi}I$$

$$\geqslant 8\pi^2(V_0+V_1)-\frac{2}{3}\pi(F_0M_1+F_1M_0)$$

$$-\frac{2}{3}\{2F_0F_1[3(N_0F_1+N_1F_0)-4\pi(F_0+F_1)-4M_0M_1]\}^{1/2},$$

the equality holds if and only if ∂K_0, ∂K_1 are congruent spheres. We have completed the proof.

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References

[1] Santrló, L. A., Integral Geometry and Geometric Probability, Addison-Wesley Publishing Company, 1976.

[2] Ren Delin, An Introduction to Integral Geometry (In chinese, to appear).

[3] Su Buqing, Hu Hesheng, etc., Differential Geometry, 1979.

[4] Hadwiger, H., Uberdeckung ebener Bereiche durch Kreise und Quadrate, *Comment. Math. Helr.*, **13** (1941), 195—200.

[5] Hadr, H., Gegenseitige Bedeckbarkeit Zweier Eibereiche und Isoperimetrie, *Vierteljschr. Naturforsch. Gesellsch. Zürich*, **86** (1941), 152—156.