Chin. Ann. of Math.
9B (4) 1988

ON A CONJECTURE OF F. NEVANLINNA CONCERNING DEFICIENT FUNCTION (II)

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Abstract

The paper proves on the basis of [1] the following theorem: Let $f(z)$ be an entire function of lower order $\mu<\infty$, and $a_l(z)(l=1, 2, \cdots, k)$ be meromorphic functions which satisfy $T(r, a_l(z))=o\{T(r, f)\}$. If

$$\sum_{l=1}^{k}\delta(a_l(z), f)=1, \quad (a_l(z)\not\equiv\infty), \tag{1}$$

then the deficiencies $\delta(a_l(z), f)$ are equal to $\frac{n_l}{\mu}$, where n_l is an integer, $l=1, 2, \cdots, k$.

§1. Introduction

In 1930, it was conjectured by F. Nevanlina [2] that if $f(z)$ is a meromorphic function of order $\lambda<\infty$ and assume $\sum_a \delta(a, f)=2$, then: a) λ is of the form $n/2$, where n is a positive integer; b) $\nu(f)\leqslant 2\lambda$, where $\nu(f)$ is the number of deficient values of $f(z)$; and c) each of the deficiencies is of the form $\delta(a, f)=k_i/\lambda$, where k_i is an interger.

In 1946, Pfluger[3] proved the conjecture when f is an entire function. In 1982, Li Qing-Zhong and Ye Ya-Sheng[4] proved that if (1) holds, then $\lambda=\mu$ and λ is a positive integer. We prove in this paper the con jecture c) is true under the same assumptions.

§2. The Proof of The Theorem

Since $f(z)$ is an entire function satisfying (1), it follows from [1] that $\nu(f)\leqslant\mu$, and then $f(z)$ has k $(k\leqslant\mu)$ deficient functions $a_1(z), a_2(z)\cdots\cdots, a_k(z)$.

Let $a_1(z), a_2(z), \cdots, a_h(z)$ be the largest linear indepedent system and denote

$$L(f)=\begin{vmatrix} f(z), & a_1(z), & \cdots, & a_h(z) \\ f'(z), & a_1'(z), & \cdots, & a_h'(z) \\ \cdots & \cdots & \cdots & \cdots \\ f^{(h)}(z), & a_1^{(h)}(z), & \cdots, & a_l^{(h)}(z) \end{vmatrix}.$$

Manuscript received May 10, 1986.

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We have known from [4] that both $L(f)(z)$ and $f(z)$ are regular growth functions and the order of $f(z)$ is equal to the order of $L(f)(z)$. And from [5] we have obtained

$$\lim_{r\to\infty}\frac{N(r,\ (L(f))^{-1})}{T(r,f)}=0,\quad \lim_{r\to\infty}\frac{T(r,\ L(f))}{T(r,f)}=1. \tag{2}$$

Noticing that $N(r,\ L(f))=o\{T(r,\ f)\}$, we get

$$K\{L(f)\}=\overline{\lim_{r\to\infty}}\frac{N(r,\ L(f))+N(r,\ (L(f))^{-1})}{T(r,\ L(f))}=0.$$

Now we take $p=\lambda(=\mu)$ in [6] Theorem 1 and Lemma 5 and thus $L(f)$ satisfies the assumptions in the two theorems. It follows that

$$T(r,\ L(f))=\frac{|C(r)|r^{\lambda}}{\pi}(1+o(1)), \tag{3}$$

$$|C(\sigma r)-C(r)|=o\{C(r)\},\quad 1<\sigma\leqslant 36, \tag{4}$$

$$T(\sigma r,\ L(f))=(1+o(1))\sigma^{\lambda}\cdot T(r,\ L(f)), \tag{5}$$

$$|\log|L(f)|-\mathrm{Re}\{C_j z^{\lambda}\}|=o\{|C_j|r^{\lambda}\},\quad z\in\Gamma_j-E_j, \tag{6}$$

where $C(r)=\alpha_0+\frac{1}{\lambda}\{\sum_{|a_\nu|\leqslant\nu}a_\nu^{-\lambda}-\sum_{|b_\mu|<r}b_\mu^{-\lambda}\}$ and $a_\nu,\ b_\mu$ are the zeros and poles of $f(z)$ respectively, $C_j=C(\alpha^j)$, $\alpha=\exp((\lambda+1)^{-1})$, Γ_j is the annulus: $\alpha^j\leqslant|Z|<\alpha^{j+3/2}$, E_j consists of finite number of circles and the sum of radius of these circles is no more then $4e\ \delta\alpha^{j+2}$, where δ is a sufficiently small positive number.

Proceed on the same way as in [1]: We select $\rho_j\in[\alpha^j,\ \alpha^{j+5/4}]$, so that the circle $|Z|=\rho_j$ does not intersect E_j, set $C_j=|C_j|^{iw_j}$. We divide $|z|=\rho_j$ into 2λ arcs such that $\cos(\lambda\theta+w_j)\leqslant 0$ on λ arcs $\alpha_1^j,\ \cdots,\ \alpha_\lambda^j$ and $\cos(\lambda\theta+w_j)\geqslant 0$ on λ arcs $\beta_1^j,\ \cdots,\ \beta_\lambda^j$. Obviously, α_ν^j and β_ν^j are apart from each other and the anglar measure of each α_ν^j and β_ν^j is π/λ. We denote by $\alpha_\nu^j(\theta)$, $\beta_\nu^j(\theta)$ the sets

$$\alpha_\nu^j(\theta)=\{\theta;\ z=\rho_j\colon e^{i\theta}\in\alpha_\nu^j\}=[\phi_\nu^j,\ \phi_\nu^j+\pi/\lambda]\ \text{(assume)}$$

$$\beta_\nu^j(\theta)=\{\theta;\ z=\rho_j\in\beta_\nu^j\}.$$

With no loss of generality we assume $0<\phi_1^j<\cdots<\phi_\lambda^j<2\pi$.

Set $z=re^{i\theta}$ in what follows. Since $\cos(\lambda\theta+w_j)\geqslant 0$ for $\theta\in\beta_\nu^j(\theta)$ $(\nu=1,\ 2,\ \cdots,\ \lambda)$, it follows from (6) that

$$\log|f(z)-a_l(z)|+\log\left|\frac{L(f)}{f(z)-a_l(z)}\right|$$
$$=\log|L(f)|\geqslant|C_j|r^{\lambda}\cos(\lambda\theta+w_j)+o\{|C_j|r^{\lambda}\}\geqslant o\{|C_j|r^{\lambda}\}.$$

Hence

$$\log^+\frac{1}{|f(z)-a_l(z)|}=\log^+\left|\frac{L(f)}{f(z)-a_l(z)}\right||+o\{|C_j|r^{\lambda}\}\quad(1\leqslant l\leqslant k). \tag{7}$$

Since $\alpha_j\leqslant\rho_i<3\alpha_j$, we have from (2), (3), (4)

$$\frac{|C_j|\rho_j^{\lambda}}{T(\rho_j,f)}=\frac{|C_j|}{|C(\rho_j)|}\cdot\frac{|C(\rho_j)|\rho_j^{\lambda}}{T(\rho_j,\ L(f))}\cdot\frac{T(\rho_j,\ L(f))}{T(\rho_j,f)}\to\pi\ (j\to\infty). \tag{8}$$

Combining (7), (8), we have for $\nu=1,\ 2,\ \cdots,\ \lambda$; $l=1,\ 2,\ \cdots,\ k$

$$\overline{\lim_{j\to\infty}} \frac{1}{2\pi T(\rho_j, f)} \int_{\beta_\nu^j} \log^+ \frac{1}{|f(\rho_j e^{i\theta}) - a_l(\rho_j e^{i\theta})|} d\theta$$
$$\leqslant \overline{\lim_{j\to\infty}} \left\{ \frac{1}{2\pi T(\rho_j, f)} \int_{\beta_\nu^j} \log^+ \left| \frac{L(f)(\rho_j e^{i\theta})}{f(\rho_j e^{i\theta}) - a_l(\rho_j e^{i\theta})} \right| d\theta + \frac{o\{|C_j| \rho_j^\lambda\}}{T(\rho_j, f)} \right\} = 0. \tag{9}$$

Put $d^j_{\nu,l} = \frac{1}{2\pi T(\rho_j, f)} \int_{\alpha_\nu^j} \log^+ \left| \frac{1}{f(\rho_j e^{i\theta}) - a_l(\rho_j e^{i\theta})} \right| d\theta$ ($\nu = 1, 2, \cdots, \lambda$; $l = 1, 2, \cdots, k$). Then form (9) we derive for each l ($l = 1, 2, \cdots, k$),

$$\delta(a_l(z), f) = \underline{\lim_{j\to\infty}} \frac{1}{2\pi T(r, f)} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a_l(re^{i\theta})|} d\theta$$
$$\leqslant \underline{\lim_{j\to\infty}} \left\{ \frac{1}{2\pi T(\rho_j, f)} \sum_{1=\nu}^{\lambda} \int_{\alpha_\nu^j} \log^+ \frac{1}{|f(\rho_j e^{i\theta}) - a_l(\rho_j e^{i\theta})|} d\theta \right.$$
$$= \underline{\lim_{j\to\infty}} \sum_{\nu=1}^{\lambda} d^j_{\nu,l}. \tag{10}$$

When $l=1$, for each, j, we rearrange $d^j_{\nu,l}$ in increasing order. Since the number permutation of $d^j_{\nu,l}$ ($\nu = 1, 2, \cdots, \lambda$) is finite, there exists a subsequence $\{j_1\}$ of $\{j\}$, which corresponds to a permutation $(\nu_{11}, \nu_{12}, \cdots, \nu_{1\lambda})$ of $(1, 2, \cdots, \lambda)$, such that $d^{j_1}_{\nu_{11},1} \geqslant d^{j_1}_{\nu_{12},1} \geqslant \cdots \geqslant d^{j_1}_{\nu_{1\lambda},1}$. In the same way, we can obtain a subsequence $\{j_2\}$ of $\{j_1\}$ and a corresponding permtation $(\nu_{21}, \nu_{22}, \nu_{2\lambda})$ of $(1, 2, \cdots, \lambda)$, such that $d^{j_2}_{\nu_{12},2} \geqslant d^{j_2}_{\nu_{22},2} \geqslant \cdots \geqslant d^{j_2}_{\nu_{2\lambda},2}$, and by the same method we can get a subsequence $\{j_k\}$ of $\{j\}$, without loss the generality, still denoted by $\{j\}$, such that

$$\begin{cases} d^j_{\nu_{11},1} \geqslant d^j_{\nu_{12},1} \geqslant \cdots \geqslant d^j_{\nu_{1\lambda},1}, \\ d^j_{\nu_{21},2} \geqslant d^j_{\nu_{22},2} \geqslant \cdots \geqslant d^2_{\nu_{2\lambda},2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ d^j_{\nu_{k1},k} \geqslant d^j_{\nu_{k2},k} \geqslant \cdots \geqslant d^j_{\nu_{k\lambda},k}, \end{cases}$$

where $(\nu_{l1}, \nu_{l2}, \cdots, \nu_{lk})$ is a permutation of $(1, 2, \cdots, \lambda)$. Let $D_{\nu_{ln},l} = \underline{\lim_{j\to\infty}} d^j_{\nu_{ln},l}$ ($l=1, 2, \cdots, k$; $n = 1, 2, \cdots, \lambda$). It follows obviously from (9) that

$$D_{\nu_{l1},l} > 0 \text{ and } D_{\nu_{l1},l} \geqslant D_{\nu_{l2},l} \geqslant \cdots \geqslant D_{\nu_{l\lambda},l} \geqslant 0 \quad (l = 1, 2, \cdots, k).$$

Denote by D_l the smallest non-zero one of $D_{\nu_{ln},l}$. Obviously $D_l = D_{\nu_{lN_l},l} > 0$. Set

$$D = \min\{D_1, D_2, \cdots, D_k\}, \quad (>0).$$

Then $d^j_{\nu_{ln},l} > D/2 > 0$, $l = 1, 2, \cdots, k$; $n = 1, 2, \cdots, N_l$, provided j is chosen to be sufficiently large. Denote by $\alpha^j_{\nu_{ln},l}(\theta) = [\phi^j_{\nu_{ln},l}, \phi_{\nu_{ln},l} + \pi/\lambda]$ the angular set corresponding to $d^j_{\nu_{ln},l}$. We take η properly small and $I^j_{\nu_{ln},l} = [\phi^j_{\nu_{ln},l} + \eta, \phi^j_{\nu_{ln},l} + \pi/\lambda - \eta]$, then $I^j_{\nu_{ln},l} \subset \alpha_{\nu^j_{ln},l}(\theta)$. Furthermore, by (10), the (3.19) of [1] holds, that is, for arbitrary z, $z = \rho_j e^{i\theta}$, $\theta \in I^j_{\nu_{ln},l}$, we have

$$|f(z) - a_l(z)| \leqslant \exp(-CT(\rho_j, f)), \quad l = 1, 2, \cdots, \cdots, k,$$

where $C>0$. We can prove in the same way as in [1] that, if $l \neq \tilde{l}$ ($1 \leqslant l, \tilde{l} \leqslant k$), $I^j_{\nu_{l\bar{n}},l}$ and $I^j_{\nu_{\tilde{l}\tilde{n}},\tilde{l}}$ are apart from each other ($\bar{n} = 1, 2, \cdots, N_l$; $\tilde{n} = 1, 2, \cdots, N_{\tilde{l}}$). Then there exist $\left(\sum_{l=1}^{k} N_l\right)$ $I^j_{\nu_{ln},l}$ which are apart from each other, and each $I^j_{\nu_{ln},l}$ corresponds to only one $\alpha^j_{\nu_{ln},l}$. Noticing that the number of $\alpha^j_{\nu_{ln},l}$ is no more than λ, we have

$$N_1+N_2+\cdots+N_k\leqslant\lambda. \tag{11}$$

We now prove $\delta(a_l(z), f)\leqslant N_c/\lambda$. $(l=1, 2, \cdots, k)$.

We have from (6) for $\varepsilon>0$

$$\log|L(f)|\geqslant|C_j|r^\lambda\cos(\lambda\theta+w_j)-o\{|C_j|r^\lambda\},$$

when $\theta\in\alpha^j_{\nu_{ln},l}$ $(n=1, 2, \cdots, N_l)$, $z=re^{i\theta}$, that is

$$\log^+\frac{1}{|L(f)|}\leqslant-|C_j|r^\lambda\cos(\lambda\theta+w_j)+o\{|C_j|r^\lambda\}.$$

Combining the above inequality with (8) we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\alpha^j_{ln,l}}\log^+\frac{1}{|f(\rho_je^{i\theta})-a_l(\rho_je^{i\theta})|}d\theta\\
&\leqslant\frac{1}{2\pi}\int_{\alpha^j_{\nu_{ln},l}}\log^+\left|\frac{|L(f)(\rho_je^{i\theta})}{f(\rho_je^{i\theta})-a_l(\rho_je^{i\theta})}\right|d\theta+\frac{1}{2\pi}\int_{\alpha^j_{\nu_{ln},l}}\log^+\frac{1}{|L(f)(\rho_je^{i\theta})|}d\theta\\
&\leqslant o\{T(\rho_j, f)\}-\frac{|C_j|\rho_j^\lambda}{2\pi}\int_{\alpha^j_{\nu_{ln},l}}\cos(\lambda\theta+w_j)d\theta+o\{|C_j|\rho_j^\lambda\}\\
&=\frac{1}{\lambda\pi}|C_j|\rho_j^\lambda+o\{T(\rho_j, f)\}\quad(n=1, 2, \cdots, N_l;\ l=1, 2, \cdots, k).
\end{aligned}$$

Then there exists a subsequence of $\{j\}$, which without loss of generality is still denoted by $\{j\}$. Thus for $l=1, 2, \cdots, k$, we get $\underline{\lim}_{j\to\infty} d^j_{\nu_l(N_l+1)},_l=0$, and $\overline{\lim}_{j\to\infty} d^j_{\nu_{ln}},_l\leqslant1/\lambda$, $n=1, 2, \cdots, N_l$; $l=1, 2, \cdots, k$. Since we have assumed that

$$D_{\nu_l(N_l+1)},_l=\underline{\lim}_{j\to\infty}d^j_{\nu_l(N_l+1)},_l=0,$$

$$\lim_{j\to\infty}d^j_{\nu_l(N_l+2)},_l=\lim_{j\to\infty}d^j_{\nu_l(N_l+3)},_l=\cdots=\lim_{j\to\infty}d^j_{\nu_{l\lambda}},_l=0.$$

Then, from (9), and noticing that (9) holds for any subsequence of $\{j\}$, we obtain

$$\delta(a_l(z), f)\leqslant\underline{\lim}_{j\to\infty}\sum_{n=1}^{\lambda}d^j_{\nu_{ln}},_l\leqslant\overline{\lim}_{j\to\infty}\sum_{n=1}^{N}d^j_{\nu_{sn}},_l\leqslant\frac{N_l}{\lambda},\quad(l=1, 2, \cdots, k).$$

Combining (1) with (11), it follows that

$$1=\sum_{l=1}^{k}\delta(a_l(z), f)\leqslant\frac{N_1+N_2+\cdots+N_k}{\lambda}\leqslant1.$$

Then we must have $\delta(a_l(z), f)=\dfrac{N_l}{\lambda}$, $l=1, 2, \cdots, k$. We complete our proof.

The authors would like to express their sincere thanks to Professor Li Ruifu and Professor Yang Le for their guidance.

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