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LARGE TIME STEP GENERALIZATIONS OF GLIMM'S SCHEME FOR SYSTEMS OF CONSERVATION LAWS**

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Abstract

Two kinds of generalizations of Glimm's scheme for Courant numbers larger than $\frac{1}{2}$ is introduced. For one kind of the generalizations, referred to as L.T.S. Glimm's scheme (I), it is shown that for any fixed (but arbitrary large) Courant number if a sequence of approximate solutions converges to a limit u as the mesh is refined, then u is a werk solution to the system of conservation laws/for almost choise of random sequence. Furthermore it is obtained that for scalar equations and systems of conservation laws the family of approximate solutions contains convergent subsequnce.

For another kind of generalizations with any fixed (but arbitrary large) Courant number, referred to as L. T. S. Glimm's scheme (II), it is proved that the family of approximate solutions to the system of isothermal gas dynamics equations contains a convergent subsequence provided the total variation of the initial data is bounded.

§1. Introduction

We are concerned with systems of conservation laws of the form

$$u_t+f(u)_x=0, \quad -\infty<x<\infty, \ t\geqslant 0 \tag{1.1}$$

with initial conditions

$$u(x, 0)=u_0(x), \quad -\infty<x<\infty. \tag{1.2}$$

Here $u(x, t)=(u^1(x, t), \cdots, u^m(x, t))\in R^m$ and $f^1(u(x, t)), \cdots, f^m(u(x, t)))$ is a smooth mapping from a region Ω of R^m to R^m. We assume that the system (1.1) is strictly hyperbolic in the sence that the matrix $\frac{\partial f}{\partial u}$ has real and distinct eigenvalues

$$\lambda_1(u)<\lambda_2(u)<\cdots<\lambda_m(u).$$

Solutions to the initial value problem (1.1), (1.2) generally develop discontinuities (shocks) even when the initial data $u_0(x)$ is smooth. Therefore we seek weak solutions to (1.1), (1.2), i. e., solutions $u(x, t)$ which satisfy

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$$\int_0^\infty\int_{-\infty}^\infty (u\phi_t+f(u)\phi_x)\,dx\,dt+\int_{-\infty}^\infty u_0(x)\phi(x,0)\,dx=0 \tag{1.3}$$

for any C^1 test function $\phi(x, t)$ with compact support.

The major breakthrough in the theory of systems of conservation laws is Glimm's work [3], in which a random choice difference scheme, Glimm's scheme, is introduced to construct the approximate solution to the initial value problem (1.1), (1.2) and the convergence and consistency of the scheme is obtained provided $\vee u_0(\cdot)$ is sufficienlly small and system is genuinely nonlinear. Here throughout this paper by $\overset{b}{\underset{a}{\vee}} g(\cdot)$ ($\vee g(\cdot)$ resp.) we mean the total variation of $g(x)$ over the interval (a, b) ($(-\infty, \infty)$ resp.). It is fair to say that most of the subsequent work on theory of systems of conservation laws is based on it. The Glimm's scheme is also an effective method for numerical calculations even in several space variables by the method of fractional steps[1]. The main advantages of Glimm's scheme for numerical calculations are sharp resolution of discontinuities and absence of over shoots and undershoots.

Now we give a brief introduction to the Glimm's scheme. Before doing it we assume that the Riemann problems for systems (1.1) with suitable Riemann data can be solved into unique centered wave solutions. We discretize $R\times[0, \infty)$ by choosing spatial mesh length δ and time mesh length τ. Let $t_n=n\tau$, $n=0, 1, 2, \cdots$, $x_k=k\delta$, $x_{k\pm1/2}=\left(k\pm\frac{1}{2}\right)\delta$, $k=0, \pm1, \pm2\cdots$. We choose a random sequence $\{\alpha_i\}$ equidistributed in (0, 1). Assuming that $u(x, t, \delta)$, an approximate solution to initial value problem, has been determined for $t<t_n$, we define $u(x, t_{n'}, \delta)=u_{k,n}=u(x_k+\alpha_n\delta, t_n-0)$ for $x\in I_k=(x_k, x_{k+1})$. Let $u(x, t, \delta)$ on $Y_{k,n}=\{(x, t), x_{k-1/2}<x<x_{k+1/2}, t_n\leqslant t<t_{n+1}\}$ be the restriction on $Y_{k,n}$ of the solution $u_{k,n}(x, t)$ to the Riemann problem:

$$\begin{cases}(1.1)\\ u(x, t_n)=\begin{cases}u_{k-1,n}, & x<x_k\\ u_{k,n}, & x>x_k.\end{cases}\end{cases} \tag{1.4}$$

To initiate the scheme, at $n=0$, we set

$$u(x, 0-, \delta)=u_0(x).$$

Each wave in the solution to Riemann problem travels with speed that equals or is bounded by cooresponding eigenvalue λ_i. Denote $|\lambda|=\max\limits_{1\leqslant i\leqslant m}|\lambda_i(u)|$ for all u under consideration, then it follows that the centered waves issuing from adjacent centers (x_k, t_n) and (x_{k+1}, t_n) do not intersect each other provided

$$\frac{\tau}{\delta}|\lambda|=C\leqslant\frac{1}{2}. \tag{1.5}$$

Therefore $u(x, t, \delta)$ in the strip $S_n=\{(x, t), -\infty<x<\infty, t_n\leqslant t<t_{n+1}\}$ is a

concatenation of solutions to Riemann problems (1.4) at (x_k, t_n), $k=0, \pm1, \pm2, \cdots$, thus $u(x, t, \delta)$ is a weak solution to system (1.1) in S_n. Thus (1.5) is always assumed for Glimm's scheme. Here C is called Courant number defined as mesh ratio τ/δ multiplied by $|\lambda|$.

Since it is a time consuming procedure to solve Riemann problem and that Courant number is not larger than 1/2 is only a sufficient condition for the hold of Courant–Friedrichs–Lewy condition, it would be reasonable to generatize Glimm's scheme to Courant number larger than 1/2. In fact, LeVeque[5] has successfully generalized Godunov's method for arbitrarily large Courant number in proving the consistency of the large time step generalization of Godunov's method for system of conservation laws and the convergence for scalar conservation laws.

In this paper a large time step genralization of Glimm's scheme, which, for short, we refer to as L. T. S. Glimm's scheme (I), is introduced in section 2 and we show the consistency of the L. T. S. Glimm's scheme (I) for any fixed (but arbitrarily large) Courant number.

In section 3 we prove that the total variations of the approximate solutions, constructed by L. T. S. Glimm's scheme (I), to initial value problems for scalar conservation laws and general systems (1.1) with corresponding initial data are uniformly bounded. Therefore by the consistency of L. T. S. Glimm's scheme (I) obtained in section 2 the existence of global solutions to these initial value problems is proved. At the same section another large time step generatization of Glimm's scheme, which is called L. T. S. Glimm's scheme (II) for short, is described, and the converngence of this scheme for the system of isothermal gas dynamics equations is obtained.

§2. A Large Time Step Generalization of Glimm's Scheme and Its Consistency

When Courant numbers are larger than 1/2, the centered waves issuing from adjacent centers may interact each other in S_n. Of course it would be not practical or even impossible to handle these interactions exactly. We should handle these interactions properly and approximately so that approximate results of these interactions are accurate enough to make L. T. S. Glimm's schemes consistency and convergence as well as the procedure of handling these interactions as simple as possible for numerical calculations. LeVeque's work [5] suggests we may handle wave interactions linearly in some sense.

Now we give descriptions of the L. T. S. Glimm's scheme (I). Discretizing $R\times[0, \infty)$ we allow the Courant number C to be arbitrary (fixed) constant, i. e.,

$$\frac{\tau}{\delta}|\lambda| = C \leqslant N, \tag{2.1}$$

where N is the least integer no smaller than C. After determination of the approximate solution $u(x, t, \delta)$ for $t<t_n$, we solve the Riemann problem (1.4) at mesh points (x_k, t_n), $k=0, \pm1, \pm2, \cdots$ and obtain their solutions $u_{k,n}(x, t) = (u^1_{k,n}(x, t), \cdots, u^m_{k,n}(x, t))$. In order not to be overloaded with subscripts we omit the subscript n in the sequel. Then we set the approximate solution as follows:

$$u(x, t, \delta) = u_k(x, t) + \sum_{i \neq k}(u_i(x, t) - u_i(x, t_n)), \quad (x, t) \in Y_{k,n}, \tag{2.2}$$

or the same

$$u(x, t, \delta) = u(x, t_n, \delta) + \sum_i (u_i(x, t) - u_i(x, t_n)), \quad (x, t) \in S_n. \tag{2.3}$$

The above procedure may proceed for all $t>0$ provided we have suitable bounds on $u(x, t, \delta)$. We assume these for the moment and demonstrate the consistency of L. T. S. Glimm's scheme (I):

Proposition 2.1. *Assume that each choice of the random sequence $\{\alpha_i\}$ yields a family $\{u(x, t, \delta), 0<\delta<\delta_0)$ of approximate solutions which are defined and $\bigvee u(\cdot, t, \delta)$ are uniformly bounded in δ and t. Then there exists a sequence $\delta_l \to 0$ such that for almost all choices of $\{\alpha_i\}$*

$$u(x, t, \delta_l) \xrightarrow{\text{a. e}} u(x, t), \quad \delta_l \to 0$$

and $u(x, t)$ is the solution to the initial value problem (1.1), (1.2).

Proof First we prove that our approximate solutions are L_1 Lipschitz continuous in time (modulo the time step). Let $x \in I_k$, $t' \in [t_{n_1}, t_{n_1+1})$ and $t'' \in [t_{n_2}, t_{n_2+1})$, $n_1 \leqslant n_2$. Then by (2.3) we have for $1 \leqslant j \leqslant m$

$$\begin{aligned} |u^j(x, t'', \delta) - u^j(x, t', \delta)| &\leqslant |u^j(x, t'', \delta) - u^j(x, t_{n_2}, \delta)| \\ &\quad + |u^j(x, t_{n_2}, \delta) - u^j(x, t_{n_2}-, \delta)| + \cdots + |u^j(x, t_{n_1+1}, \delta) - u^j(x, t_{n_1+1}-, \delta)| \\ &\quad + |u^j(x, t_{n_1+1}-, \delta) - u^j(x, t', \delta)| \\ &\leqslant 2 \sum_{n_1 \leqslant l \leqslant n_2} \bigvee_{y_{k-N}}^{x_{k+N+1}} u^j(\cdot, t_l+, \delta). \end{aligned} \tag{2.4}$$

Therefore

$$\begin{aligned} \int_{-\infty}^{\infty} |u^j(x, t'', \delta) - u^j(x, t', \delta)| dx &\leqslant 2(2N+1)(n_2 - n_1 + 2)\delta \sup_{t>0} \bigvee u^j(\cdot, t, \delta) \\ &\leqslant \frac{(4N+2)}{C}|\lambda|(t'' - t' + 3\tau) \sup_{t>0} \bigvee u^j(\cdot, t, \delta). \end{aligned} \tag{2.5}$$

The inequalities (2.4), (2.5) and the uniform boundedness of $\nabla u(\cdot, t, \delta)$ imply that the approximate solutions are L_1 Lipschitz continuous in time (modulo the time step), which with boundedness of $\bigvee u(\cdot, t, \delta)$ yields that there exists a sequence $\delta_l \to 0$ such that

$$u(x, t, \delta_l) \xrightarrow{\text{a. e}} u(x, t), \quad \delta_l \to 0.$$

Now we are going to show that $u(x, t)$ is the (weak) solution to the initial value problem (1.1), (1.2). For any test function $\phi(x, t)$ we have the following obvious identities:

$$\int_0^\infty\int_{-\infty}^\infty (u(x, t, \delta)\phi_t+f(u(x, t, \delta))\phi_x)dx\,dt+\int_0^\infty u_0(x)\varphi(x, 0)dx$$

$$=I+\sum_{\substack{k=0,\pm1,\pm2\cdots\\ n=0,1,2\cdots}} I_{k,n}, \tag{2.6}$$

where

$$I=\sum_{n=0}^\infty\int_{-\infty}^{+\infty}(u(x, t_n-, \delta)-u(x, t_n, \delta))\phi(x, t_n)dx, \tag{2.7}$$

$$I_{k,n}=\iint_{Y_{k,n}}(u(x, t, \delta)\phi_t+f(u(x, t, \delta))\phi_x)dx\,dt$$

$$+\int_{\partial Y_{k,n}} u(x, t, \delta)\phi dx-f(u, t, \delta))\phi dt, \tag{2.8}$$

and

$$I^i_{k,n}=\iint_{Y_{kn}}(u_i(x, t)\phi_t+f(u_i(x, t))\phi_x)dx\,dt+\int_{\partial Y_{k,n}} u_i(x, t)\phi\,dx-f(u_i(x, t))\phi\,dt$$

$$=0, \quad k-N\leqslant i\leqslant k+N, \tag{2.9}$$

$$\bar I^i_{k,n}=\iint_{Y_{kn}}(u_i(x, t_n)\phi_t+f(u_i(x, t_n))\phi_x)dx\,dt+\int_{\partial Y_{k,n}}(u_i(x, t_n)\phi\,dx$$

$$-f(u_i(x, t_n))\phi\,dx)=0, \quad k-N\leqslant i\leqslant k+N,\ i\neq k. \tag{2.10}$$

Here that $u_i(x, t)$, $k-N\leqslant i\leqslant k+N$ and $u_i(x, t_n)$, $k-N\leqslant i\leqslant k+N$, $i\neq k$ are weak solutions on $Y_{k,n}$ is used. It follows from (2.8), (2.9) and (2.10) that

$$I_{k,n}=I_{k,n}-\sum_{k-N\leqslant i\leqslant k+N} I^i_{k,n}+\sum_{\substack{i\neq k\\ n-N\leqslant i\leqslant k+N}}\bar I^i_{k,n}=I'_{k,n}+I''_{k,n}, \tag{2.11}$$

where

$$I'_{k,n}=\iint_{Y_{kn}}[f(u(x, t, \delta))-f(u_k(x, t))-\sum_{\substack{i\neq k\\ k-N\leqslant i\leqslant k+N}}(f(u_i(x, t))-f(u_i(x, t_n)))]\cdot\phi_x dx\,dt, \tag{2.12}$$

$$I''_{k,n}=\int_{t_n}^{t_{n+1}}[f(u(x, t, \delta))-f(u_k(x, t))$$

$$-\sum_{\substack{i\neq k\\ k-N\leqslant i\leqslant k+N}}(f(u_i(x, t))-f(u_i(x_i, t_n)))]\,|_{x_{k-1/2}}^{x_{k+1/2}}\cdot\phi\,dt. \tag{2.13}$$

Here (2.2) is used.

By mean value theorem we have

$$|I'_{k,n}|=\Big|\iint_{Y_{kn}}\int_0^1\frac{\partial f}{\partial u}(u_k(x, t)+\theta(u(x, t, \delta)-u_k(x, t)))\cdot(u(x, t, \delta)-u_k(x, t)).$$

$$d\theta\,\phi_x dx\,dt-\iint_{Y_{k,n}}\sum_{\substack{i\neq k\\ k-N\leqslant i\leqslant k+N}}\int_0^1\frac{\partial f}{\partial u}(u_i(x, t_n)+\theta(u_i(u, t)-u_i(x, t_n)))$$

$$\times(u_i(x, t)-u_i(x, t_n))d\theta\,\phi_x dx\,dt\Big|$$

$$\leqslant\iint_{Y_{k,n}} 2|\phi_x|\left|\frac{\partial f}{\partial u}\right|\sum_{\substack{k-N\leqslant i\leqslant k+N\\ i\neq k}}|u_i(x,t)-u_i(x, t_n)|\,dx\,dt$$

$$\leqslant \nu \iint_{Y_{k,N}} \bigvee_{x_{k-N}}^{x_{k+N}} u(\cdot,\ t_n+,\ \delta)\,dx\,dt = \nu\tau\delta \bigvee_{x_{k-N}}^{x_{k+N}} u(\cdot,\ t_n+,\ \delta), \tag{2.14}$$

where the constant ν depends on the L_∞ norms of ϕ_x and the matrix $\frac{\partial f}{\partial u}$, which is bounded for smoothness of f and uniform boundedness of $u(x,\ t,\ \delta)$.

Noting that

$$u_i(x_{k\pm1/2}-,\ t) = u_i(x_{x\pm1/2}+,\ t),\quad t\in(t_n,\ t_{n+1})$$

except for possible finite points of $(t_n,\ t_{n+1})$ and $u_n(x_{k\pm1/2}-,\ t_n) = u_i(x_{k\pm1/2}+,\ t_n)$ for $k=0,\ \pm1,\ \pm2,\ \cdots,\ n=0,\ 1,\ 2,\ \cdots$, we obtain

$$\sum_k I''_{k,n} = 0. \tag{2.15}$$

Thus it follows from (2.11), (2.14) and (2.15) that

$$\Big|\sum_k I_{k,n}\Big| \leqslant 2N\,\nu\tau\delta \bigvee_{-\infty}^{\infty} u(\cdot,\ t_n+,\ \delta). \tag{2.16}$$

The number of s_n in which the test function ϕ does not vanish identically is order of τ^{-1} since ϕ has compact support. Therefore we obtain from (2.16)

$$\Big|\sum_{k,n} I_{k,n}\Big| \leqslant \delta\bar{\nu} \sup_{t>0} \bigvee_{-\infty}^{\infty} u(\cdot,\ t,\ \delta), \tag{2.17}$$

where $\bar{\nu}$ depends on N, ϕ and L_∞ norm of matrix $\frac{\partial f}{\partial u}$.

By Glimm's celebrated argument there exists a subsequence of $\{u(x,\ t,\ \delta_l)\}$ (denoted by $\{u(x,\ t,\ \delta_l)\}$ again) such that I tends to zero as $\delta_l \to 0$ for almost all choices of $\{a_i\}$. Thus it follows from (2.6) and (2.17) that

$$\int_0^\infty \int_{-\infty}^{+\infty} (u(x,\ t)\phi_t + f(u(x,\ t))\phi_x)\,dx\,dt + \int_{-\infty}^{+\infty} u_0(x)\phi(x,\ 0)\,dx = 0$$

i. e. $u(x,\ t)$ is a weak solutisn to initial value problem (1.1), (1.2). The proof is complete.

§3. Convergence of Large Time Step Generalizations of Glimm's Scheme

In this section we establish the bounds on the total variations of the approximate solutions constructed by L. T. S. Glimm's scheme (I) for scalar conservation laws and the general systems (1.1) with corresponding suitable initial data. Then by Proposition 2.1, the global solutions to the initial value problems are obtained as the limits of some sequences of these approximate solutions when the mesh is refined. Another large time step generalization of Glimm's scheme, called L. T. S. Glimm's scheme (II), will be described and the uniform bound on the total variations of the approximate solutions, constructed by L. T. S. Glimm's scheme (II), to the system of isothermal gas dynamics equations with arbitrary intial data of bounded total variations is also obtained.

First we consider the initial value problem for scalar conservation law

$$u_t+f(u)_x=0, \tag{3.1}$$

$$u(x, 0)=u_0(x), \tag{3.2}$$

where $f\in C^1$ and $u_0(x)$ is an arbitrary function of bounded toatal variation.

Lemma 3.1. *$\vee u(\cdot, t, \delta)$, the total variation of the approximate solutions to initial value problem* (3.1), (3.2) *constructed by L. T. S. Glimm's scheme* (I), *is uniformly bounded in t and δ.*

Proof Since the centered wave solution to the Riemann problem (1.4) for scalar equation takes on its value monotonely from $u_{k-1,n}$ to $u_{k,n}$[2], we obtain

$$\sum_k \vee u_k(\cdot, t)=\sum_k |u_{k,n}-u_{k-1,n}|\leqslant \vee u(\cdot, t_n-, \delta), \quad t\in S_n. \tag{3.3}$$

It follows from (2.3) that

$$\begin{aligned} &u(x+y, t, \delta)-u(x, t, \delta)\\ &\quad=[(u(x+y, t_n, \delta)-u(x, t_n, \delta))-\sum_k(u_k(x+y, t_n)-u(x, t_n))]\\ &\qquad+\sum_k(u_k(x+y, t)-u_k(x, t)), \quad t\in S_n. \end{aligned} \tag{3.4}$$

As in [5] we know that if $0<y<\delta$ then the term in the braces is zero. Hence

$$u(x+y, t, \delta)-u(x, t, \delta)=\sum_k(u_k(x+y, t)-u_k(x, t)), \quad y<\delta, \ t\in S_n, \tag{3.5}$$

$$|u(x+y, t, \delta)-u(x, t, \delta)|\leqslant\sum_k|u_k(x+y, t)-u_k(x, t)|, \quad y<\delta, \ t\in S_n, \tag{3.6}$$

which implies

$$\vee u(\cdot, t, \delta)\leqslant\sum_k \vee u_k(\cdot, t), \quad t\in S_n. \tag{3.7}$$

It is easy to know inductively from (3.3) and (3.7) that

$$\vee u(\cdot, t, \delta)\leqslant\vee u(\cdot, t_n-, \delta)\leqslant \vee u_0(\cdot), \quad t\in S_n. \tag{3.8}$$

The proof is complete.

Proposition 2.1 and Lemma 3.1 yield the following theorem.

Theorem 3.2. *The family $\{u(x, t, \delta), 0<\delta<\delta_0\}$ of approximate solutions to initial value problem* (3.1), (3.2) *constructed by L. T. S. Glimm's scheme* (I) *contains a subsequence $u(x, t, \delta_l)$ such that for almost all choices of $\{a_i\}$*

$$u(x, t, \delta_l) \xrightarrow{\text{a. e}} u(x, t), \quad \delta_l\to 0$$

and $u(x, t)$ is the solution to the initial value problem (3.1), (3.2).

We now introduce another large time step generalization of Glimm's scheme, L. T. S. Glimm's scheme (II). Without loss of generality we may assume that for system (1.1) there exists a local coordinate system $w(u)=(w^1(u), \cdots, w^m(u))$ detined in Ω consisting of Riemann invariants, i. e.

$$r_i\cdot\nabla w^j=\delta_{ij}, \quad 1\leqslant i, j\leqslant m,$$

where r_i is the right eigenvector of $\frac{\partial f}{\partial u}$ for λ_i. In fact there exists such a coordinate system for $m=2$ and while studing general systems there exists a coordinate system $(w^1(u), \cdots, w^m(u))$ having the property $r_j\nabla w^i(\bar{u})=\delta_{ij}$, $1\leqslant i, j\leqslant m$, where $\bar{u}$ is a constant state near which we consider the system[3].

The approximate solutions $u(x, t, \delta)$ to the initial value problem (1.1), (1.2) are constructed by L. T. S. Glimm's scheme (II) as follows:

The discretization of $R\times[0, \infty)$ is done as in L. T. S. Glimm's scheme (I) so that (2.1) holds. After determinating the approximate solution $u(x, t, \delta)$ for $t<t_n$, we solve the Riemann problem (1.4) at mesh points (x_k, t_n), $k=0, \pm1, \pm2, \cdots$, and obtain their solutions $u_k(x, t)$. Then we set

$$u(x, t, \delta)=(u^1(w(x, t, \delta)), \cdots, u^m(w(x, t, \delta))), \quad (x, t)\in S_n, \tag{3.9}$$

where

$$w(x, t, \delta)=w_k(x, t)+\sum_{i\neq k}(w_i(x, t)-w_i(x, t_n)), \quad (x, t)\in Y_{k,n} \tag{3.10}$$

or the same

$$w(x, t, \delta)=w(u(x, t_n, \delta))+\sum_i(w_i(x, t)-w_i(x, t_n)), \quad (x, t)\in S_n \tag{3.11}$$

and

$$w_k(x, t)=(w^1(u_k(x, t)), \cdots, w^m(u_k(x, t))), \ (x, t)\in S_n, \ k=0, \pm1, \pm2\cdots. \tag{3.12}$$

Next we consider initial value problem of the system of isothermal gas dynamics equations

$$\begin{cases} u_t+(v^{-1})_x=0, \\ v_t-u_x=0, \end{cases} \tag{3.13}$$

with initial data

$$(u, v)|_{t=0}=(u_0(x), v_0(x)), \tag{3.15}$$

where u is velocity and v is specific volume of the gas, $u_0(x)$ and $v_0(x)$ are arbitrary functions of bounded total variation with $v_0(x)\geqslant \underline{v}>0$.

The existence of global solutions to initial value problem (3.13), (3.14) was proved by Nishida[6] via Glimm's scheme. Now we shall prove the convergence of the approximate solution, constructed by L. T. S. Glimm's scheme (II), to initial value problem (3.13), (3.14).

Theorem 3.3. *The family* $\{(u, v)(x, t, \delta), 0<\delta<\delta_0\}$ *of approximate solutions to the initial value problem* (3.13), (3.14) *constructed by L. T. S. Glimm's scheme* (II) *contains a sequence* $(u, v)(x, t, \delta_l)$ *such that*

$$(u, v)(x, t, \delta_l)\xrightarrow{\text{a. e}}(u, v)(x, t), \quad \delta_l\to 0.$$

Proof The Riemann invariants for system (3.13) are $w_i=u+(-1)^i q$, $i=1, 2$, where $q=-\log v$.

The Riemann problem (3.13) with the Riemann data

$$(u, v)(x, 0)=\begin{cases}(u_l, v_l), & x<0, \\ (u_r, v_r), & x>0,\end{cases}$$

where u_l, u_r, $v_l>0$ and $v_r>0$ are constants, has a piecewise continuous solution. This solution consists of at most three constant states (u_l, v_l), (u_m, v_m) and (u_r, v_r) separated by a 1-wave (1-shock or 1-rarefaction wave) and a 2-wave (2-shock or 2-rarefaction wave) respectively. We define the functional $Q((u_l, q_l), (u_r, q_r)) = |q^l - q^m| + |q_m - q_r|$, the sum of the absolute values of differences of q across the two waves in above solution, which is the tolal variation of q in this solution too. By the fact[6] that shock and rarefaction wave curves in terms of (u, q) have the same figures respectively and are independent of thier initials, we have for there arbitrary constant states (u_-, q_-), (n_0, q_0) and (u_+, q_+)

$$Q((u_-, q_-), (u_+, q_+)) \leqslant \Omega((u_-, q_-), (u_0, q_0)) + Q((u_0, q_0), (u_+, q_+)). \quad (3.15)$$

From the fact that the mapping between (w_1, w_2) and (u, q) is linear, (3.11) is equivalent to

$$(u, q)(x, t, \delta) = (u, q)(x, t_n, \delta) + \sum_i ((u_i, q_i)(x, t) - (u_i, q_i)(x, t_n)), \quad (x, t) \in S_n, \quad (3.16)$$

where u_i, $q_i = -\log v_i$ is the solution to (3.13) with Riemann data

$$(u, v)(x, t_n) = \begin{cases} (u, v)(x_{i-1} + \alpha_n\delta, t_n -, \delta), & x < x_i, \\ (u, v)(x_i + \alpha_n\delta, t_n -, \delta), & x > x_i. \end{cases} \quad (3.17)$$

Proceeding as we obtain (3.5) in Lemma 3.1 we can show

$$\begin{aligned} &(u, q)(x+\xi, t, \delta) - (u, q)(x, t, \delta) \\ &= \sum_i ((u_i, q_i)(x+\xi, t) - (u_i, q_i)(x, t)), \quad (x, t) \in S_n. \end{aligned} \quad (3.18)$$

By repeated applying (3.15) to both sides of (3.18) we obtain for $t = t_{n+1}$, $x = x_k + \alpha_{n+1}\delta$, $x + \xi = x_{k-1} + \alpha_{n+1}\delta$

$$\begin{aligned} &Q((u, q)(x_{k-1} + \alpha_{n+1}\delta, t_{n+1}^-, \delta), (u, q)(x_k + \alpha_{n+1}\delta, t_{n+1}^-, \delta)) \\ &\leqslant \sum_i Q((u_i, q_i)(x_{k-2} + \alpha_{n+1}\delta, t_{n+1}^-), (u_i, q_i)(x_k + \alpha_{n+1}\delta, t_{n+1}^-)). \end{aligned} \quad (3.19)$$

Summing the both sides in (3.19) over k we obtain

$$\bigvee q(\cdot, t, \delta) \leqslant \sum_i \bigvee q_i(\cdot, t) \leqslant \sum_i \bigvee q_i(\cdot, t_{n+1}-) = \sum_i \bigvee q_i(\cdot, t_n+) = \bigvee q(\cdot, t_n+, \delta),$$
$$t \in (t_{n+1}, t_{n+2}).$$

Then inductively

$$\bigvee q(\cdot, t, \delta) \leqslant \bigvee q_0(0, t) \leqslant \frac{1}{\underline{v}} \bigvee v_0(\cdot) + \bigvee u_0(\cdot), \quad t > 0. \quad (3.20)$$

Hence

$$\sup_{(x,t) \in R \times [0, \infty)} |q(x, t, \delta)| \leqslant M_0, \quad (3.21)$$

where

$$M_0 = |\log v_0(-\infty)| + \frac{1}{\underline{v}} \bigvee v_0(\cdot) + \bigvee v_0(\cdot),$$

$$e^{-M_0} \leqslant \sup_{(x,t) \in R \times [0, \infty)} \bigvee (x, t, \delta) \leqslant e^{M_0}, \quad (3.22)$$

$$\bigvee v(\cdot, t, \delta) \leqslant e^{M_0} \bigvee q(\cdot, t, \delta) \leqslant \frac{M_0 e^{M_0}}{\underline{v}}, \quad t>0, \tag{3.23}$$

$$\bigvee u(\cdot, t, \delta) \leqslant e^{M_0} \bigvee v(\cdot, t, \delta) \leqslant \frac{M_0 e^{2M_0}}{\underline{v}}, \quad t>0. \tag{3.24}$$

It follows from (3.22) that $v(x, t, \delta)$ has positive lower bound. Therefore $(u, v)(x, t, \delta)$ can be defined for all $t>0$. As in the proof of Proposition 2.1, we can prove that $(u, v)(x, t, \delta)$ are L_1 Lipschitz continuous in time, which with (3.23) and (3.24) yield that there exists a sequence $(u, v)(x, t, \delta_l)$ such that

$$(u, v)(x, t, \delta_l) \xrightarrow{\text{a. e}} (u, v)(x, t).$$

The proof is complete.

Finally we return to consider initial value problem (1.1), (1.2). Let $r_i(u)$ and $l_i(u)$ be the right and left eigenvectors of $\frac{\partial f}{\partial u}$ for the eigenvalue λ_i. In addition to strict hyperbolicity we assume that the system (1.1) is genuinely nonlinear in the sense that

$$r_j \cdot \nabla\lambda_j \neq 0, \quad 1 \leqslant j \leqslant m.$$

Near constant state $\tilde{u}$ we normalize r_j, l_j such that

$$r_j \nabla\lambda_j = 1, \quad 1 \leqslant j \leqslant m \tag{3.25}$$

and then

$$l_j \cdot r_j = 1, \quad 1 \leqslant j \leqslant m. \tag{3.26}$$

Furthermore we assume that each eigenvalue does not change its sign, i. e. there exists an index, say l, such that for u near $\tilde{u}$

$$\lambda_l(u) < 0 \leqslant \lambda_{l+1}(u). \tag{3.27}$$

We recall some well known definitions and results (see [3, 4, 7]). If u_l and u_r are any two constant states near $\tilde{u}$, then the Riemann problem (1.1) with initial data

$$u(x, 0) = \begin{cases} u_l, & x<0, \\ u_r, & x>0 \end{cases} \tag{3.28}$$

which, for short, we refer to as Riemann problem (u_l, u_r), has unique centered wave solution, denoted by (u_l, u_r), consists of $m+1$ constant states $u_0 = u_l, u_1, \cdots, u_{m-1}, u_m = u_r$, where u_{k-1} is connected on the right to u_k by a k- wave (a k- shock or a k-rarefaction wave). We choose w_i $1 \leqslant i \leqslant m$, defined near $\tilde{u}$ such that

$$r_j \nabla w_j = 1, \quad 1 \leqslant j \leqslant m \tag{3.29}$$

and

$$r_j \cdot \nabla w_k |_{\tilde{u}} = 0, \quad j \neq k. \tag{3.30}$$

Then we define

$$s_j = w_j(u_j) - w_j(u_{j-1}), \quad 1 \leqslant j \leqslant m \tag{3.31}$$

as the signed strength of j-wave. We shall gather all this information in the concise form

$$(u_l, u_r) = [(u_0, u_1, \cdots, u_m)/(\varepsilon_1, \cdots, \varepsilon_m)]. \tag{3.32}$$

We have the expansion for (u_l, u_r)

$$u_r - u_l = \sum_{1\leqslant i\leqslant m} \varepsilon_i r_i(u_l) + \sum_{1\leqslant h\leqslant i\leqslant m} \varepsilon_i \varepsilon_h E_{i,h}(u_l, u_r), \tag{3.33}$$

where $E_{i,h}(u_l, u_r)$ is Lipschitz continuous in u_l and u_r, $1\leqslant h\leqslant i\leqslant m$. Let u_l, u_m, u_r be three constant states near u and let

$$(u_l, u_m) = [(u_0', \cdots, u_m')/(r_1, \cdots, r_m)],$$
$$(u_m, u_r) = [(u_0'', \cdots, u_m'')/(\delta_1, \cdots, \delta_m)] \tag{3.34}$$

denote the solutions of respective Riemann problems (u_l, u_m) and (u_m, u_r). Glimm proved that

$$|\varepsilon_i| < |\gamma_i| + |\delta_i| + CD(\gamma, \delta), \quad \text{as } |\gamma| + |\delta| \text{ small}, \tag{3.35}$$

where $D(\gamma, \delta) = \sum|\gamma_i||\delta_i|$, the sum is over all pairs for which the i-wave in u' and j-wave in u'' are approaching. In what follows C denotes the generic constant depending on flux function f and some small neighborhood of $\tilde{u}$.

Let u_l, u_r, $\bar{u}_l$ and $\bar{u}_r$ be four constant states near $\tilde{u}$ and

$$(u_l, u_r) = [(u_0, \cdots, u_m)/(\varepsilon_1, \cdots, \varepsilon_m)],$$
$$(\bar{u}_l, \bar{u}_r) = [(\bar{u}_0, \cdots, \bar{u}_m)/(\bar{\varepsilon}_1, \cdots, \bar{\varepsilon}_m)],$$

and suppose that

$$|u_r - u_l| \leqslant C|\varepsilon_i|, \quad |\bar{u}_r - \bar{u}_l| \leqslant C|\bar{\varepsilon}_j|, \tag{3.36}$$

which mean that the strength of i-wave (j-wave, resp.) in (u_l, u_r) ($(\bar{u}_l, \bar{u}_r)$ resp.) is of the same order of $|\varepsilon|$ ($|\bar{\varepsilon}|$ resp.). Then we have the following lemma.

Lemma 3.4. *For $(u_l+\Delta, u_r+\Delta) = [(u_0', \cdots, u_m')/(\varepsilon_1', \cdots, \varepsilon_m')]$ of the solution of Riemann problem $(u_l+\Delta, u_r+\Delta)$ we have the folliwing estimates*

$$|\varepsilon_k' - \varepsilon_k| \leqslant C|\varepsilon_i||\bar{\varepsilon}_j|, \quad 1\leqslant k\leqslant m, \tag{3.37}$$

where $\Delta = \bar{u}_r - \bar{u}_l$.

Proof Applying the expansion (3.33) for (u_l, u_r) and $(u_l+\Delta, u_r+\Delta)$ we obtain

$$(u_r - u_l) = \sum_i \varepsilon_i r_i(u_l) + \sum_{h\leqslant g} \varepsilon_g \varepsilon_h E_{g,h}(u_l, u_r), \tag{3.38}$$

$$(u_r - u_l) = \sum_i \varepsilon_i' r_i(u_l+\Delta) + \sum_{h\leqslant g} \varepsilon_g' \varepsilon_h' E_{g,h}(u_l+\Delta, u_r+\Delta). \tag{3.39}$$

Multiply both sides of (3.38) and (3.39) by $l_k(u_l)$ and $l_k(u_l+\Delta)$ respectively and subtract each other, then we have

$$\begin{aligned}\varepsilon_k' - \varepsilon_k = {} & (l_k(u_l+\Delta) - l_k(u_l))(u_r - u_l) \\ & - \sum_{h\leqslant g} (l_k(u_l+\Delta)\cdot E_{g,h}(u_l+\Delta, u_r+\Delta)\varepsilon_g'\varepsilon_h' - l_k(u_l)\cdot E_{g,h}(u_l, u_r)\varepsilon_g\varepsilon_h)\end{aligned} \tag{3.40}$$

and therefore

$$\begin{aligned}|\varepsilon_k' - \varepsilon_k| \leqslant {} & C|\Delta||u_r - u_l| + \sum_{h\leqslant g} |l_k(u_l+\Delta)\cdot E_{g,h}(u_l+\Delta, u_r+\Delta) \\ & - l_k(u_l)\cdot E_{g,h}(u_l, u_r)||\varepsilon_g||\varepsilon_k| \\ & + \sum_{h\leqslant g} |l_k(u_l+\Delta) E_{g,h}(u_l+\Delta, u_r+\Delta)\cdot(\varepsilon_g'\varepsilon_h' - \varepsilon_g\varepsilon_h)|.\end{aligned} \tag{3.41}$$

It follows from (3.36), (3.41) that

$$|\varepsilon_k'-\varepsilon_k|=C|\varepsilon_i||\bar{\varepsilon}_j|+C\sum_{h\le g}|\bar{\varepsilon}_i||\varepsilon_g||\varepsilon_h|+C\sum_{h\le g}(|\varepsilon_h||\varepsilon_g-\varepsilon_g'|+|\varepsilon_g'||\varepsilon_h-\varepsilon_h'|). \tag{3.42}$$

Then by (3.36) and simple calculations we reach

$$|\varepsilon_k'-\varepsilon_k|=C|\varepsilon_i||\bar{\varepsilon}_j|, \quad 1\le k\le m. \tag{3.43}$$

The proof is complete.

In what follows we assume that $N=1$, i. e.

$$\frac{\tau}{\delta}|\lambda|\le 1. \tag{3.44}$$

First we define a functional for our approximate solutions $u(x, t, \delta)$ as follows

$$F(t_n)=L(t_n)+KQ(t_n), \tag{3.45}$$

where

$$L(t_n)=\sum\{|\alpha|:\ \alpha \text{ crosses the line } t=t_n+\frac{\delta}{2}|\lambda|^{-1}\}, \tag{3.46}$$

$$Q(t_n)=\sum\left\{|\alpha||\beta|:\alpha \text{ and } \beta \text{ cross the line } t=t_n+\frac{\delta}{2}|\lambda|^{-1} \text{ and approach}\right\}, \tag{3.47}$$

$$K=4mC, \tag{3.48}$$

C is the generic constant appearing in estimates (3.35) and (3.37).

Lemma 3.5. *If $\vee u_0(\cdot)$ is small enough such that*

$$12mCL(t_0)\le 1. \tag{3.49}$$

Then

$$L(t_i)\le 2L(t_0), \quad i=1, 2\cdots, \tag{3.50}$$

which implies that $\vee u(\cdot, t, \delta)$ is uniformly bounded in t and δ and $u(x, t, \delta)$ are defined for all $t>0$.

Proof At the line $t=t_n$, we arrange all of the discontinuous points of $u(x, t_n-0, \delta)$ and $\{x_k+a_n\delta,\ k=0, \pm 1, \pm 2\cdots\}$ in increasing order (and rename them) as follows

$$X=\{\cdots<x'_{-(k+1)}<x'_{-k}<\cdots<x'_{-1}<x'_0<x'_1<\cdots<x'_k<x'_{k+1}<\cdots\}.$$

Then we construct an imaginary solution $\bar{u}(x, t, \delta)$ for $(x, t)\in X_n=\{(x, t): t_n\le t\le t_n(x),\ x\in R\}$ as a concatenation of the solutions to the Riemann problems (1.1) at the following points of the line $t=t_n$

$$\cdots<x'_{-(k+1)}<x''_{-(k+1)}<x'_{-k}<\cdots<x'_0<\cdots<x'_k<x''_k<x'_{k+1}$$

with Riemann data

$$\bar{u}(x, t_n)=\begin{cases}u(x_k'-, t_n-, \delta), & x<x_k',\\ u(x_k'+, t_n-, \delta), & x>x_k'\end{cases}$$

and

$$\bar{u}(x, t_n)=\begin{cases}u(x_k'+, t_n-, \delta) & x<x_k'',\\ u(x_{k+1}'-, t_n-, \delta) & x>x_k'',\end{cases}$$

where

$$x_k'' = \frac{x_k' + x_{k+1}'}{2},$$

and $t=t_n(x)$ is a curve such that there is no interaction between waves from $\bar{u}(x, t, \delta)$ on x_n.

Then we define the functional for $\bar{u}(x, t, \delta)$ as follows

$$\bar{F}(t_n) = \bar{L}(t_n) + K\bar{Q}(t_n), \tag{3.45'}$$

where

$$\bar{L}(t_n) = \sum\{|\alpha|, \ \alpha \text{ crosses the curve } t=t_n(x)\}, \tag{3.46'}$$

$$\bar{Q}(t_n) = \sum\{|\alpha||\beta|, \ \ \alpha \text{ and } \beta \text{ cross the curve } t=t_n(x) \text{ and approach}\}. \tag{3.47'}$$

Suppose that

$$L(t_i) \leqslant 2L(t_0), \quad i \leqslant n-1, \tag{3.51}$$

which is true for $i=1$.

Then we study the approximate solution $u(x, t, \delta)$ defined by (2.3) in S_{n-1}. Proceeding as we obtain (3.5) in Lemma 3.1, we can obtain

$$u(x+\eta, t, \delta) - u(x, t, \delta) = \sum_k (u_k(x+\eta, t) - u_k(x, t)),$$

$$(x, t) \in S_{n-1}. \tag{3.52}$$

Specially for a discontinuous point $(x', t') \in S_{n-1}$

$$u(x'-, t', \delta) - u(x'+, t' \delta) = \sum_k (u_k(x'-, t) - u_k(x'+, t)). \tag{3.53}$$

A shock, say, a j-shock S_j, in $u(x, t, \delta)$ will become a discontinuous line $S_j'(\bar{t})$ after interacting on other waves and discontinuous lines. It follows from (3.53) that $S_j'(\bar{t})$ has the same speed of propagation and the same jump in terms of u as S_j does. On the other hand, by (2.3) we know that the difference of u of left states (right states resp.) between the shock S_j and the discontinuous line $S_j'(\bar{t})$ equals to the algebraic sum of jumps of u in the waves and discontinuous lines which have interacted with S_j or S_j' before $t=\bar{t}$. Therefore the Hugoniot jump condition does not hold for $S_j'(\bar{t})$ in general. This means that the left state of $S_j'(\bar{t})$ can not be connected on the right by a j-shock to right state of $S_j'(\bar{t})$. Similarly a rarefaction wave curve after moving without changing its shape in term of u becomes a curve which is not a rarefaction wave curve in general also. But Lemma 3.4 gives the quantitative estimates on this phenomena. Therefore using Lemma 3.4 at each point (x_k', t_n), (x_k'', t_n), $k=0, \pm1, \pm2, \cdots$, we obtain from (3.46), (3.46)', (3.47), (3.47)'

$$\bar{L}(t_n) - L(t_{n-1}) \leqslant 2mCP(t_n), \tag{3.54}$$

$$\bar{Q}(t_n) - Q(t_{n-1}) = 2mCP(t_n)L(t_{n-1}) + (2mCP(t_n))^2 - P(t_n), \tag{3.55}$$

where

$$P(t_n) = \sum\{|\alpha||\beta| : \alpha \text{ and } \beta \text{ in } u(x, t, \delta) \text{ cross the line}$$

$$t = t_{n-1} + \frac{\delta}{2}|\lambda|^{-1} \text{ and will pass each other before } t=t_n\}. \tag{3.56}$$

Here (3.44) and (3.27) are used to guarantee that there is no interation between waves belonging the same family in S_{n-1}.

It follows from (3.54), (3.55) that

$$\begin{aligned}\bar{F}(t_n)-F(t_{n-1})&\leqslant 2mCP(t_n)+KP(t_n)(2mCL(t_{n-1})+(2mCL(t_{n-1}))^2-1)\\ &\leqslant 2mCP(t_n)+KP(t_n)(3mCL(t_{n-1})-1\\ &\leqslant 2mCP(t_n)-\frac{K}{2}P(t_n)\leqslant 0,\end{aligned}\tag{3.57}$$

where (3.51) and (3.48) are used.

Since $\{x_k+\alpha_n\delta,\ k=0,\ \pm1,\ \pm2\cdots\}$ is a subset of X, we know that

$$F(t_n)\leqslant\bar{F}(t_n),\tag{3.58}$$

which with (3.57) implies

$$F(t_n)\leqslant F(t_{n-1}).\tag{3.59}$$

Therefore inductively we prove that

$$L(t_n)\leqslant F(t_n)\leqslant F(t_{n-1})\leqslant\cdots\leqslant F(t_0)\leqslant 2L(t_0),\quad n=1,\ 2\cdots.\tag{3.60}$$

The inequalities yield that

$$\vee u(\cdot,\ t_n+,\ \delta)\leqslant C\vee u_0(\cdot),\quad n=1,\ 2\cdots.\tag{3.61}$$

It follows from (3.52) that

$$\vee u(\cdot,\ t,\ \delta)\leqslant\vee u(\cdot,\ t_n+,\ \delta),\ t\in S_n,\quad n=1,\ 2\cdots.\tag{3.62}$$

It is easy to know that

$$\sup_x|u(x,\ t,\ \delta)|\leqslant|u_0(-\infty)|+\vee u(\cdot,\ t,\ \delta)\leqslant|u_0(-\infty)|+C\vee u_0(\cdot).\tag{3.63}$$

Thus $\{\lambda_j(x,\ t,\ \delta),\ 1\leqslant j\leqslant m\}$ is uniformly bounded in x, t and δ. Hence $u(x,\ t,\ \delta)$ can be defined for all $t>0$. The proof is completed.

Lemma 3.5. and Proposition 2.1. yield the following theorem.

Theorem 3.6. *If* $\vee u_0(\cdot)$ *is sufficiently small, then the family* $\{u(x,\ t,\ \delta),\ 0<\delta<\delta_0\}$ *of approximate solutions to the initial value problem* (1.1), (1.2) *constructed by L. T. S. Glimm's scheme* (I) *contains a sequence* $(u,\ v)(x,\ t,\ \delta_l)$ *such that for almost all choices of* $\{a_i\}$

$$u(\delta,\ t,\ \delta_l)\xrightarrow{\text{a. e}}u(x,\ t),\ \delta_l\to 0,$$

and $u(x,t)$ *is the solution to the initial value problem* (1.1), (1.2).

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