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ON THE CONVERGENCE OF THE PADÉ TABLE

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Abstract

This paper offers a general result on the problem whether there exists an infinite subsequence of $[L/M](z)$ approximants from the $(M+1)$-th row of the Padé table converging to $f(z)$. According to this result one can get the optimum conclusion (mentioned in the last paragraph).

For a given formal power series

$$f(z)=f_0+f_1z+f_2z^2+\cdots \tag{1}$$

and for a pair of non-negative integers L and M, the Padé approximant $[L/M](z)$ to $f(z)$ is defined to be the rational function $[L/M](z)=\frac{P_{L/M}(z)}{Q_{L/M}(z)}$ satisfying the formal identity

$$Q_{L/M}(z)f(z)-P_{L/M}(z)=cz^{L+M+1}+\text{terms of higher degree}, \tag{2}$$

where $P_{L/M}(z)$ and $Q_{L/M}(z)$ are polynomials of degrees not exceeding L and M respectively, $Q_{L/M}(z)\not\equiv 0$ and c is a constant. The Padé table is the double infinite array of $[L/M](z)$,

$$\begin{matrix} [0/0](z) & [1/0](z) & [2/0](z) & \cdots \\ [0/1](z) & [1/1](z) & [2/1](z) & \cdots \\ [0/2](z) & [1/2](z) & [2/2](z) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

The row of $[L/M]$ Padé approximants is the sequence

$$[0/M](z),\ [1/M](z),\ [2/M](z),\ \cdots. \tag{3}$$

One of the main problem about the convergence of rows of the Padé table is whether there exists an infinite subsequence of $[L/M]$ approximants from the $(M+1)$-th row of the Padé table converging within the largest circle centered at the origin which contains not more than M poles of a given function and within which the function is meromorphic[1,2]. In 1983 [4] gave a special counterexample. Here the author proves a general result as follows.

Theorem. *For any positive number r, and for non-negative integers M and u with $u\geqslant 2$, there exists a meromorphic function $f(z)$ that is analytic in the disk $|z|<r$*

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except for M non-zero poles $\{z_n\}_{n=1}^M$, counted according to multiplicity. No subsequence of the row of $[L/M+u]$ Padé approximants converges to $f(z)$ on every compact subset of $\{z: |z|<r\}\backslash\{z_n\}_{n=1}^M$ uniformly.

Proof Suppose $r=1$ without loss of generalization. First let $M=0$. For integer u and non-zero complex number s satisfying $u\geqslant 2$ and $u+1+s\neq 0$, define a function

$$f(z)=\frac{s}{(u+1+s)(1-z)}+\frac{1}{1-z_1z}+\frac{1}{1-z_2z}+\cdots+\frac{1}{1-z_uz}, \tag{4}$$

where 1, z_1, z_2 $\cdots$, z_u are the $u+1$ roots of the equation $z^{u+1}-1=0$, and

$$z_t=\exp\left(\frac{2\pi t i}{u+1}\right),\quad 1\leqslant t\leqslant u.$$

Obviously $f(z)$ is analytic in $|z|<1$. Let $f(z)=\sum_{L=0}^{\infty}f_Lz^L$. Then

$$f_L=\begin{cases}-\dfrac{u+1}{u+1+s}, & \text{if } L\not\equiv 0(\operatorname{mod}(u+1)),\\ \dfrac{(u+1)(u+s)}{u+1+s}, & \text{if } L\equiv 0(\operatorname{mod}(u+1)),\end{cases}\qquad (L=0,1,2,\cdots). \tag{5}$$

Using Jacobi's formula to $[L/M](z)$ [3]a, we have

$$Q_{L/u}(z)=\pm\frac{(u+1)^u}{u+1+s}\cdot B_p(z,s), \tag{6}$$

where $L\equiv p\ (\operatorname{mod}(u+1))$, $B_p(z,s)=(z^u+z^{u-1}+\cdots+1)+sz^p$, $(p=0,1,2,\cdots,u)$. For present purpose we have to show that there exists a complex number s such that every $B_p(z,s)$ possesses zeros in $|z|<1$. For $p=u$, we choose s to satisfy

$$-\frac{\pi}{2}<\arg s<\frac{\pi}{2},$$

then $|1+s|>1$, and the polynomial

$$B_u(z,s)=(1+s)\left(z^u+\frac{1}{1+s}z^{u-1}+\cdots+\frac{1}{1+s}\right)$$

has zeros in $|z|<1$. For each fixed $p(p=0,1,\cdots,u-1)$, let $B_p(z,s)=\prod_{t=1}^{u}(z-x_t)$, each x_t approaching z_t as $s\to 0$. From the definition of z_t, we have $\prod_{t=1}^{n}(z-z_t)=z^u+z^{u-1}+\cdots+1$. Therefore

$$\prod_{t=1}^{u}(z-x_t)-\prod_{t=1}^{u}(z-z_t)=sz^p. \tag{7}$$

Put $z=x_{t_p}\in\{x_t\}_{t=1}^u$ in (7) and denote $d_{t_p}=x_{t_p}-z_{t_p}$. Then

$$d_{t_p}\cdot\prod_{t\neq t_p}(x_{t_p}-z_t)=-sx_{t_p}^p. \tag{8}$$

Because $0<\dfrac{2\pi t}{u+1}<2\pi$ and $x_t\doteq z_t=\exp\left(\dfrac{2\pi t}{u+1}i\right)$ if $|s|$ is sufficiently small, we have

$$\begin{aligned}x_{t_p}-z_t\approx z_{t_p}-z_t&=\exp\left(\frac{2\pi t_p}{u+1}i\right)-\exp\left(\frac{2\pi t}{u+1}i\right)\\&=\exp\left(\frac{\pi(t_p+t)}{u+1}i\right)\cdot 2i\cdot\sin\frac{\pi(t_p-t)}{u+1}\quad (t\neq t_p),\end{aligned}$$

$$\begin{cases} \arg(x_{t_p}-z_t) \doteq +\frac{\pi}{2}+\frac{\pi(t_p+t)}{u+1}, & \text{if } t<t_p, \\ \arg(x_{t_p}-z_t) \doteq -\frac{\pi}{2}+\frac{\pi(t_p+t)}{u+1}, & \text{if } t>t_p. \end{cases} \tag{9}$$

By (8) and (9), for small $|s|$ we have

$$\arg d_{t_p} \doteq -\pi+\arg s+\frac{2\pi p t_p}{u+1}-\left[\sum_{t=1}^{t_p-1}\left(\frac{\pi}{2}+\frac{\pi(t_p+t)}{u+1}\right)+\sum_{t=t_p+1}^{u}\left(-\frac{\pi}{2}+\frac{\pi(t_p+t)}{u+1}\right)\right]$$

$$=\arg s-\frac{\pi}{2}-\frac{2(u-p)-1}{u+1}\pi t_p. \tag{10}$$

The above formula keeps valid for any fixed $x_{t_p}\in\{x_t\}_{t=1}^{u}$. Thus we have

$$\arg d_t := \arg(x_t-z_t) \doteq \arg s-\frac{\pi}{2}-\frac{2(u-p)-1}{u+1}\pi t \quad (t=1, 2, \cdots, u).$$

If we can find a pair of integers k_p and t_p $(1\leqslant t_p\leqslant u)$ such that

$$\frac{2\pi t_p}{u+1}+\frac{1.01}{2}\pi<\arg d_{t_p}+2k_p\pi<\frac{2\pi t_p}{u+1}+\frac{2.99}{2}\pi, \tag{11}$$

then

$$x_{t_p}=z_{t_p}+d_{t_p}=\exp\left(\frac{2\pi t_p}{u+1}i\right)+\delta\cdot\exp\left(\frac{2\pi t_p}{u+1}i+\theta i\right)=\exp\left(\frac{2\pi t_p}{u+1}i\right)(1+\delta e^{\theta i}),$$

where $\delta=|d_{t_p}|$ and $\frac{1.01}{2}\pi<\theta<\frac{2.99}{2}\pi$, and x_{t_p} will move into the disk $|z|<1$ if we make $|s|$ (with $\arg s$ fixed) small enough so that δ is small as well. Consolidating (10) and (11) gives the following inequalities that s should satisfy

$$\left[2\pi(1-k_p)+\frac{2(u-p)+1}{u+1}\pi t_p\right]-\frac{1.99}{2}\pi<\arg s$$

$$<\left[2\pi(1-k_p)+\frac{2(u-p)+1}{u+1}\pi t_p\right]-\frac{0.01}{2}\pi. \tag{12}$$

However, we have to find some s such that every $B_p(z, s)$ $(p=0, 1, \cdots, u)$ possesses zeros in the disk $|z|<1$. Therefore s should be chosen so that $-\frac{\pi}{2}<\arg s<\frac{\pi}{2}$ and for each p $(p=0, 1, \cdots, u-1)$ there exists a pair of integers k_p and t_p $(1\leqslant t_p\leqslant u)$ satisfying (12). If we can show that for each $p(p=0, 1, \cdots, u-1)$ there exists a pair of integers k_p and $t_p(1\leqslant t_p\leqslant u)$ satisfying

$$\frac{1.1}{2}\pi<2\pi(1-k_p)+\frac{2(u-p)+1}{u+1}\pi t_p<\frac{2.9}{2}\pi, \tag{13}$$

then (12) and (13) give

$$\frac{0.91}{2}\pi=\frac{2.9}{2}\pi-\frac{1.99}{2}\pi<\arg s<\frac{1.1}{2}\pi-\frac{0.01}{2}\pi=\frac{1.09}{2}\pi,$$

and finally we can choose some s with $\frac{0.91}{2}\pi<\arg s<\frac{\pi}{2}$ and $|s|$ small enough, the chosen s can guarantee the validity of (12) as well as of (11) for every $p(p=0, 1, \cdots, u-1)$ and satisfy the inequalities $-\frac{\pi}{2}<\arg s<\frac{\pi}{2}$. Now we have to make certain of (13).

On the assumption of $u \geqslant 2$, and for $p=0, 1, \cdots, u-1$, we have inequalities

$$\frac{3}{u+1}\pi \leqslant \frac{2(u-p)+1}{u+1}\pi \leqslant 2\pi - \frac{\pi}{u+1}. \tag{14}$$

For any fixed p ($p=0, 1, \cdots, u-1$), suppose

$$\frac{1.1}{2}\pi < \frac{2(u-p)+1}{u+1}\pi < \frac{2.9}{2}\pi. \tag{15}$$

Then (13) is true for $t_p=1$ and $k_p=1$. If (15) is not valid, we have, by (14), either

$$0 < \frac{2(u-p)+1}{u+1}\pi \leqslant \frac{1.1}{2}\pi \tag{16}$$

or

$$\frac{2.9}{2}\pi \leqslant \frac{2(u-p)+1}{u+1}\pi < 2\pi. \tag{17}$$

In the case of (16), if for every t ($t=1, 2, \cdots, u$) we have

$$0 < \frac{2(u-p)+1}{u+1}\pi t \leqslant \frac{1.1}{2}\pi,$$

then

$$0 < \frac{2(u-p)+1}{u+1}\pi u \leqslant \frac{1.1}{2}\pi,$$

and we apply the left-hand side of (14) to it and get

$$\frac{3u}{u+1}\pi \leqslant \frac{2(u-p)+1}{u+1}\pi u \leqslant \frac{1.1}{2}\pi,$$

that is, $u \leqslant \frac{1.1}{4.9}$ contrary to the assumption $u \geqslant 2$. So we assume that there exists a t_a such that

$$\begin{cases} 0 < \dfrac{2(u-p)+1}{u+1}\pi t \leqslant \dfrac{1.1}{2}\pi \quad \text{for } t=1, 2, \cdots, t_a\ (1 \leqslant t_a < u), & (18) \\ \dfrac{1.1}{2}\pi < \dfrac{2(u-p)+1}{u+1}\pi(t_a+1). & (19) \end{cases}$$

Now

$$\begin{aligned} \frac{1.1}{2}\pi &< \frac{2(u-p)+1}{u+1}\pi(t_a+1) = \frac{2(u-p)+1}{u+1}\pi t_a + \frac{2(u-p)+1}{u+1}\pi \\ &\leqslant \frac{1.1}{2}\pi + \frac{1.1}{2}\pi = \frac{2.2}{2}\pi < \frac{2.9}{2}\pi, \end{aligned} \tag{20}$$

this leads to (13) for $t_p=t_a+1$ and $k_p=1$. In the second case of (17), if for every t ($t=1, \cdots, u$) we have

$$-\frac{1.1}{2}\pi \leqslant \left(-2\pi + \frac{2(u-p)+1}{u+1}\pi\right)t < 0,$$

then

$$-\frac{1.1}{2}\pi \leqslant \left(-2\pi + \frac{2(u-p)+1}{u+1}\pi\right)u < 0$$

and we apply the right-hand side of (14) to it and get

$$-\frac{1.1}{2}\pi\leqslant\left(-2\pi+\frac{2(u-p)+1}{u+1}\pi\right)u\leqslant-\frac{\pi u}{u+1},$$

that is, $u\leqslant\frac{1.1}{0.9}$, contrary to the assumption $u\geqslant 2$. Now we assume that there exists a t_β such that

$$\begin{cases}-\frac{1.1}{2}\pi\leqslant\left(-2\pi+\frac{2(u-p)+1}{u+1}\pi\right)t<0 \quad \text{for } t=1,\,2,\cdots,\,t_\beta\ (1\leqslant t_\beta<u), & (21)\\ \left(-2\pi+\frac{2(u-p)+1}{u+1}\pi\right)(t_\beta+1)<-\frac{1.1}{2}\pi. & (22)\end{cases}$$

As before, we have

$$\begin{aligned}-\frac{2.2}{2}\pi&=-\frac{1.1}{2}\pi-\frac{1.1}{2}\pi\\ &\leqslant\left(-2\pi+\frac{2(u-p)+1}{u+1}\pi\right)t_\beta+\left(-2\pi+\frac{2(u-p)+1}{u+1}\pi\right)\\ &=\left(-2\pi+\frac{2(u-p)+1}{u+1}\pi\right)(t_\beta+1)<-\frac{1.1}{2}\pi,\end{aligned}\tag{23}$$

and (13) follows from

$$\begin{aligned}\frac{1.1}{2}\pi<\frac{1.8}{2}\pi&\leqslant 2\pi+\left(-2\pi+\frac{2(u-p)+1}{u+1}\pi\right)(t_\beta+1)\\ &=2\pi(-t_\beta)+\frac{2(u-p)+1}{u+1}\pi(t_\beta+1)<\frac{2.9}{2}\pi\end{aligned}\tag{24}$$

for $t_p=t_\beta+1=k_p$. Hence we have proved (13). By the above discussion, we see that each $B_p(z,\,s)$ $(p=0,\,1,\,\cdots,\,u)$ has a zero in $|z|<1$ for a fine s, and $[L/u](z)$ has a pole at the zero of $B_p(z,\,s)$ if $L\equiv p\ (\mathrm{mod}(u+1))$ (see (6) and Remarks behind the proof of Theorem). In the case of $M=0$, the theorem has been proved.

Secondly let $M>0$ and let

$$\hat{f}(z)=\frac{1}{1-3z}+\frac{1}{1-3y_1z}+\cdots+\frac{1}{1-3y_{M-1}z}+\frac{c_0}{1-z}+\frac{c_1}{1-z_1z}+\cdots+\frac{c_u}{1-z_uz},\tag{25}$$

where $1,\,y_1,\,y_2,\,\cdots,\,y_{M-1}$ are M roots of the equation $z^M-1=0$, $z_1,\,z_2,\,\cdots,\,z_u$ are defined as before, $c_0,\,c_1,\,\cdots,\,c_u$ are some complex numbers. Suppose $\hat{f}(z)=\sum_{j=0}^{\infty}\hat{f}_jz^j$. Then we have

$$\hat{f}_j=h_j+g_j,\ h_j=\begin{cases}0, & \text{if } j\not\equiv 0\ (\mathrm{mod}\ M),\\ 3^jM, & \text{if } j\equiv 0\ (\mathrm{mod}\ M).\end{cases}\tag{26}$$

$$g_j=c_0+c_1z_1^j+\cdots+c_uz_u^j,\quad j=0,\,1,\,2,\,\cdots.\tag{27}$$

Using $h_L=3^Mh_{L-M}$ to the denominator of $[L/M+u](z)$ of $\hat{f}(z)$ in Jacobi's formula, we get

$$Q_{L/M+u}(z)=\pm 3^{M(L-u)}[M^M(1-3^Mz^M)\cdot\Delta_L+O(3^{-L})],\tag{28}$$

where z lies in any bounded set in complex plane, and the polynomial

$$\Delta_L=\det\begin{bmatrix} z^u & \cdots & 1 \\ g_{L+M-u+1}-2\cdot 3^M g_{L-u+1}+3^{2M}g_{L-M-u+1} & \cdots & g_{L+M+1}-2\cdot 3^M g_{L+1}+3^{2M}g_{L-M+1} \\ \vdots & \ddots & \vdots \\ g_{L+M}-2\cdot 3^M g_L+3^{2M}g_{L-M} & \cdots & g_{L+M+u}-2\cdot 3^M g_{L+u}+3^{2M}g_{L-M+u} \end{bmatrix}. \tag{29}$$

Let $L\equiv 0 \pmod{(u+1)}$, and consider the following equations about g_{L-M-u}, $g_{L-M-u+1}, \cdots, g_{L+M}$

$$\begin{cases} g_{L+M-u}-2\cdot 3^M g_{L-u}+3^{2M}g_{L-M-u}=-\dfrac{u+1}{u+1+\varepsilon}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ g_{L+M-1}-2\cdot 3^M g_{L-1}+3^{2M}g_{L-M-1}=-\dfrac{u+1}{u+1+\varepsilon}, \\ g_{L+M}\ \ -2\cdot 3^M g_L\ \ +3^{2M}g_{L-M}\ \ =\dfrac{(u+1)(u+\varepsilon)}{u+1+\varepsilon}. \end{cases} \tag{30}$$

Because $g_L=g_{L^*}$ if $L\equiv L^* \pmod{(u+1)}$ (30) offers all $g_\cdot$. On the other hand, the matrix of coefficients in (30) can be rewritten so that the major terms are located in the main diagonal of the matrix. Therefore (30) has a unique group of solutions $g_0, g_1, g_2, \cdots$, which are dependent on u instead of L. With the sequence of $g_\cdot$, (27) determines $c_0, c_1, \cdots, c_u$, as its matrix of coefficients is non-singular, and polynomial Δ_L is just as $Q_{L/u}(z)$ defined by (6). According to the preceding discussion for the case $M=0$, we see that there exists a suitable ε such that the polynomial Δ_L has zeros in $|z|<1$, none of whose zeros are $\frac{1}{3}, \frac{1}{3y_1}, \cdots \frac{1}{3y_{M-1}}$. That is, for sufficiently large L, $Q_{L/M+u}(z)$ has one zero near a zero of polynomial Δ_L in the region $\{z\colon |z|<1,\ \hat{f}(z)\neq\infty\}$. Therefore no subsequence of the row of $[L/M+u](z)$ Padé approximants converges to $\hat{f}(z)$ on every compact subset of $\{z\colon |z|<1, f(z)\neq\infty\}$ uniformly. The proof is over.

Remarks. By the block construction of the Padé table[3]b, we deduce that the denominator and numerator of Padé approximant will have no common non-trivial factor if the constant term of the denominator does not vanish. In this case the zeros of the denominator are poles of the Padé approximant. This gives an auxilliary explaination for the proceeding proof.

[4] proved that a subsequence of the row of $[L/M+1]$ Padé approximants converges to $f(z)$ on any compact subset of $\{z\colon |z|<r\}\setminus\{z_n\}_{n=1}^M$ uniformly if $f(z)$ is analytic in the disk $|z|<r$ $(r>0)$ except for these M poles $\{z_n\}_{n=1}^M (z_n\neq 0)$ there. By the above theorem we conclude this result is the optimum one.

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