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ISOPARAMETRIC HYPERSURFACES IN CP^n WITH CONSTANT PRINCIPAL CURVATURES

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Abstract

This paper proves that the number of distinct principal curvatures of a real isoparametric hypersurface in CP^n with constant principal curvatures can be only 2, 3 or 5. The preimage of such hypersurface under the Hopf fibration is an isoparametric hypersurface in S^{2n+1} with 2 or 4 distinct principal curvatures. For real isoparametric hypersurfaces in CP^n with 5 distinct constant principal curvatures a local structure theorem is given.

§ 1. Preliminary

An isoparametric hypersurface M in a Riemannian manifold N is an orientable hypersurface such that each hypersurface which is obtained by "parallel translating" M along normal geodesics in N is of constant mean curvature[7]. E. Cartan proved that in constant curvature spaces isoparametric hypersurfaces are the same as the hypersurfaces with constant principal curvatures. H. F. Münzner pointed out that the number of distinct principal curvatures of an isoparametric hypersurface in the unit sphere is 1, 2, 3, 4 or 6[2].

Let CP^n denote a complex n-dimentional projective space with constant holomorphic sectional curvature 4 and π: $S^{2n+1}(\subset C^{n+1})\to CP^n$ be the Hopf fibration. If M is a (real) hypersurface in CP^n, then $\overline{M}=\pi^{-1}(M)$ is a hypersurface in S^{2n+1} and the following diagram commutes:

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\ \ i\ \ } & S^{2n+1} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\ \ i\ \ } & CP^n \end{array}$$

Let $\{e_A\}$ $(A, B, C=1, 2, \cdots, 2n)$ be a local orthonormal frame such that e_{2n} is a unit normal vector field and $\{\theta_A\}$, $\{\theta_{AB}\}$ be the dual 1-forms and connection forms respectively. Denote by $\tilde{e}_0$ the unit vertical vector field on S^{2n+1} which is

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tangent to the fibre and by $\tilde{e}_A$ the horizontal lift of e_A, i. e., $\tilde{e}_A$ are orthogonal to $\tilde{e}_0$ and $\pi_*(\tilde{e}_A)=e_A$. If $\{\tilde{\theta}_{A'}\}$, $\{\tilde{\theta}_{A'B'}\}$ $(A', B', C'=0, 1, \cdots, 2n)$ are the dual 1-forms and connection forms associated with $\{\tilde{e}_{A'}\}$, then by taking exterior differentiation of $\tilde{\theta}_A=\pi^*(\theta_A)$ and noting that π is a Riemannian submersion with totally geodesic fibres, we get the relationship between the second fundamental form $(\tilde{h}_{i'j'})$ of $\overline{M}$ and (h_{ij}) of M $(i, j=1, 2, \cdots, 2n-1;\ i', j', =0, 1, \cdots, 2n-1)$:

$$\begin{pmatrix} \tilde{h}_{ij} & \tilde{h}_{i0} \\ \tilde{h}_{0j} & \tilde{h}_{00} \end{pmatrix} = \begin{pmatrix} h_{ij} & J_{2ni} \\ J_{2nj} & 0 \end{pmatrix} \circ \pi, \tag{1.1}$$

where J_{AB} are the components of the complex structure J of CP^n (see [3]):

$$J(e_A) = -\sum J_{AB} e_B.$$

From this and the properties of Riemannian submersion that each geodesic normal to M in CP^n can be uniquely lifted to a horizontal geodesic normal to $\overline{M}$ in S^{2n+1} and π_* preserves length of horizontal verctor, we have easily the following propositions.

Proposition 1.1. *M is an isoparametric hypersurface in CP^n iff $\overline{M}$ is an isoparametric hypersurface in S^{2n+1}.*

Proposition 1.2. *Let M be a hypersurface in CP^n. Then two of the following three conditions imply the third one:*

(a) *M is isoparametric;*

(b) *M has constant principal curvatures;*

(c) *$J(e_{2n})$ is a principal vector everywhere.*

For the proof of Proposition 1.2 we refer the reader to [7] and note that by assumption of (b) and (c) we can take frame $\{e_A\}$ such that $h_{ij}=\lambda_i\delta_{ij}$ and $e_{2n-1}=-J(e_{2n})$ and (1.1) becomes

$$(\tilde{h}_{i'j'}) = \left(\begin{array}{ccc|cc} \lambda_1 & & 0 & & \\ & \ddots & & & 0 \\ 0 & & \lambda_{2n-2} & & \\ \hline & 0 & & \lambda_{2n-1} & 1 \\ & & & 1 & 0 \end{array}\right) \circ \pi. \tag{1.2}$$

This means that $\overline{M}$ is an isoparametric hypersurface in S^{2n+1} with constant principal curvatures $\lambda_1, \cdots, \lambda_{2n-2}, x_1, x_2$ where x_1, x_2 are distinct roots of equation: $x^2-\lambda_{2n-1}x-1=0$. Then by Proposition 1.1 M is isoparametric.

Basing on these Propositions we get the following theorems whose proofs will be given in next two sections.

Theorem A. *Let M be an isoparametric hypersurface in CP^n with constant principal curvatures and g the number of distinct principal curvatures. Then $g=2$, 3 or 5. $\overline{M}=\pi^{-1}(M)$ is an isoparametric hypersurface in S^{2n+1} with 2 or 4 distinct* *principal curvatures.*

This theorem is similar to H. F. Münzner's result mentioned above. From this theorem we can see that the isoparametric hypersurface $\overline{M}$ in sphere with 1, 3 or 6 distinct principal curvatures are not compatible with the Hopf fibration, or $\pi(\overline{M})$ are isoparametric hypersurfaces in CP^n with non-constant principal curvatures.

R. Takagi determined all the complete hypersurfaces in CP^n with $g=2$ or 3 distinct constant principal curvatures with Q. M. Wang's supplement for $g=3$ and $n=2$[4,5,8]. Such hypersurfaces mnst be $M_{p,q}$ or $M(2n-1, t)$ (for Definition see [4, 5]). From this together with Theorem A we get the following corollary.

Corollary. *For complete isoparametric hypersurfaces in CP^n with constant principal curvatures we have a classificatory table as follows, where g and $\bar{g}$ are the numbers of distinct principal curvatues of M and $\overline{M}$ respectively, "p. c." and "mult." are abbreviations of "principal curvatures" and "multiplicity" respectively.*

g	*p. c. of* M	*mult.*	M	$\overline{M}$	$\bar{g}$	*p. c. of* $\overline{M}$	*mult.*
2	$\operatorname{ctg} t$ $2\operatorname{ctg}(2t)$	$2n-2$ 1	$M^c_{n-1,0}$	$S^{2n-1}(\sin t)\times S^1(\cos t)$	2	$\operatorname{ctg} t$ $\operatorname{ctg}\left(t-\frac{\pi}{2}\right)$	$2n-1$ 1
3	$\operatorname{ctg} t$ $\operatorname{ctg}\left(t-\frac{\pi}{2}\right)$ $2\operatorname{ctg}(2t)$	$2p$ $2q$ 1	$M^c_{p,q}$ $p+q=n-1,\ 0<p<n-1$	$S^{2p+1}(\sin t)\times S^{2q+1}(\cos t)$	2	$\operatorname{ctg} t$ $\operatorname{ctg}\left(t-\frac{\pi}{2}\right)$	$2p+1$ $2q+1$
3	$\operatorname{ctg}\left(t-\frac{\pi}{4}\right)$ $\operatorname{ctg}\left(t-\frac{3}{4}\pi\right)$ $2\operatorname{ctg}(2t)$	$n-1$ $n-1$ 1	$M(2n-1, t)$	$\overline{M}(2n, t)$	4	$\operatorname{ctg} t$ $\operatorname{ctg}\left(t-\frac{1}{4}\pi\right)$ $\operatorname{ctg}\left(t-\frac{1}{2}\pi\right)$ $\operatorname{ctg}\left(t-\frac{3}{4}\pi\right)$	1 $n-1$ 1 $n-1$
5	$\operatorname{ctg} t$ $\operatorname{ctg}\left(t-\frac{1}{4}\pi\right)$ $\operatorname{ctg}\left(t-\frac{1}{2}\pi\right)$ $\operatorname{ctg}\left(t-\frac{3}{4}\pi\right)$ $2\operatorname{ctg}(2t)$	$2p$ m $2p$ m 1	$m+2p=n-1$		4	$\operatorname{ctg} t$ $\operatorname{ctg}\left(t-\frac{1}{4}\pi\right)$ $\operatorname{ctg}\left(t-\frac{1}{2}\pi\right)$ $\operatorname{ctg}\left(t-\frac{3}{4}\pi\right)$	$2p+1$ m $2p+1$ m

For $g=5$ in Theorem A, we have the following theorem.

Theorem B. *Let M be a complete isoparametric hypersurface in CP^n with 5 distinct constant principal curvatures. Then through each integral curve c of $J(e_{2n})$ which is a geodesic in M there are three submanifolds $M^c_{p,0}$, $M^c_{0,p}$ and $M(2m+1, t)$ $(2p+m=n-1)$ of M. Each of them is perpendicular to others, intersecting along c, and lies in a totally geodesic submanifold CP^{p+1}, CP^{p+1} or CP^{m+1} of CP^n as isoparametric hypersurface with constant principal curvatures respectively.*

§2. Proof of Theorem A

From now on we use the following convention on the ranges of indices unless otherwise stated:

$$A, B, C, \cdots=1, \cdots, n. \quad i, j, k, \cdots=1, \cdots, n-1.$$
$$\alpha, \beta, \gamma, \cdots=1, \cdots, n, \bar{1}, \cdots \overline{n-1}.$$

At first we state a known result in [1] as a lemma:

Lemma 2. 1. *Let M be a connected hypersurface in CP^n. Denote by e a unit normal vector field and A the Weingarton transformation. If on a neighborhood U of $p_0\in M$, $A(J(e))=\lambda J(e)$, then* (i) *λ is constant on U and* (ii) *if $X\in T_pM(p\in U)$ is a principal vector orthogonal to $J(e)$, $A(X)=\mu X$, then $2\mu-\lambda\neq 0$ and $A(J(X))=\dfrac{\lambda\mu+2}{2\mu-\lambda}J(X)$, i. e., $J(X)$ is also a principal vector.*

If M is a connected orientable hypersurface in CP^n, then there is a unit normal vector field e defined on M. The auther has proved in [9] that the number of distinct principal curvatures is constant implies that the multiplicity of principal curvatures are constant and thereby an orthonormal frame can be chosen so that $h_{ij}=\lambda_i\delta_{ij}$. Thus if $J(e)$ is a principal vector everywhere and the number of distinct principal curvatures is constant, then by virtue of Lemma 2.1 we can take local orthonormal frame $\{e_A, e_{\bar{A}}=J(e_A)\}$ such that $J(e_n)=e_{\bar{n}}=e$ and e_α are principal vectors, i. e.,

$$A(e_n)=\lambda_n e_n, \quad A(e_i)=\lambda_i e_i, \quad A(e_{\bar{i}})=\lambda_{\bar{i}} e_{\bar{i}}, \quad \left(\lambda_{\bar{i}}=\frac{\lambda_n\lambda_i+2}{2\lambda_i-\lambda_n}\right). \tag{2.1}$$

Let $\{\theta_A, \theta_{\bar{A}}\}$ be the dual 1-forms. By the parallelism of J the connection forms satisfy

$$\theta_{AB}=\theta_{\bar{A}\bar{B}}, \quad \theta_{A\bar{B}}=-\theta_{\bar{A}B}. \tag{2.2}$$

The Cartan structure equations for CP^n are

$$\begin{cases} d\theta_A=-\sum\theta_{AB}\wedge\theta_B-\sum\theta_{A\bar{B}}\wedge\theta_{\bar{B}}, & \theta_{AB}+\theta_{BA}=0, \\ d\theta_{\bar{A}}=-\sum\theta_{\bar{A}B}\wedge\theta_{\bar{A}B}-\sum\theta_{A\bar{B}}\wedge\theta_{\bar{B}}, & \theta_{\bar{A}\bar{B}}+\theta_{\bar{B}A}=0, \\ d\theta_{AB}=-\sum\theta_{AC}\wedge\theta_{CB}-\sum\theta_{A\bar{C}}\wedge\theta_{\bar{C}B}+\Theta_{AB}, \\ d\theta_{A\bar{B}}=-\sum\theta_{AC}\wedge\theta_{C\bar{B}}-\sum\theta_{A\bar{C}}\wedge\theta_{\bar{C}\bar{B}}+\Theta_{A\bar{B}}, \\ \Theta_{AB}=\Theta_{\bar{A}\bar{B}}=\theta_A\wedge\theta_B+\theta_{\bar{A}}\wedge\theta_{\bar{B}}, \\ \Theta_{A\bar{B}}=\Theta_{B\bar{A}}=\theta_A\wedge\theta_{\bar{B}}-\theta_{\bar{A}}\wedge\theta_B+2\delta_{AB}\sum\theta_C\wedge\theta_{\bar{C}}. \end{cases} \tag{2.3}$$

From (2.1) we know that

$$\theta_{\bar{n}i}=\lambda_i\theta_i=-\theta_{\bar{n}\bar{i}}, \quad \theta_{n\bar{i}}=\lambda_{\bar{i}}\theta_{\bar{i}}=\theta_{ni}, \quad \theta_{\bar{n}n}=\lambda_n\theta_n. \tag{2.4}$$

The structure equations for M are

$$\begin{cases} d\theta_\alpha=-\sum\theta_{\alpha\beta}\wedge\theta_\beta, \quad \theta_{\alpha\beta}+\theta_{\beta\alpha}=0, \\ d\theta_{\alpha\beta}=-\sum\theta_{\alpha\gamma}\wedge\theta_{\gamma\beta}+\Omega_{\alpha\beta}. \end{cases} \tag{2.5}$$

The Gauss equation is

$$\Omega_{\alpha\beta}=\lambda_\alpha\lambda_\beta\theta_\alpha\wedge\theta_\beta+\Theta_{\alpha\beta}. \tag{2.6}$$

By (2.2) and (2.4) we have

$$\theta_{ij}=\theta_{\bar{i}\bar{j}}, \quad \theta_{i\bar{j}}=-\theta_{\bar{i}j}, \quad d\theta_n=\sum_i(\lambda_i+\lambda_{\bar{i}})\theta_i\wedge\theta_{\bar{i}}. \tag{2.7}$$

Exterior differentiating (2.4) gives the Codazzi equations

$$\begin{cases} d\lambda_i\wedge\theta_i+\sum_j(\lambda_j-\lambda_i)\theta_{ij}\wedge\theta_j+\sum_j(\lambda^{\bar{j}}-\lambda_i)\theta_{i\bar{j}}\wedge\theta_{\bar{j}}=(1+\lambda_n\lambda_{\bar{i}}-\lambda_i\lambda_{\bar{i}})\theta_{\bar{i}}\wedge\theta_n, \\ d\lambda_{\bar{i}}\wedge\theta_{\bar{i}}+\sum_j(\lambda_j-\lambda_{\bar{i}})\theta_{\bar{i}j}\wedge\theta_j+\sum_j(\lambda_{\bar{j}}-\lambda_{\bar{i}})\theta_{\bar{i}\bar{j}}\wedge\theta_{\bar{j}}=-(1+\lambda_n\lambda_i-\lambda_i\lambda_{\bar{i}})\theta_i\wedge\theta_n, \\ \lambda_n\sum_j(\lambda_j+\lambda_{\bar{j}})\theta_j\wedge\theta_{\bar{j}}=2\sum_j(\lambda_j\lambda_{\bar{j}}-1)\theta_j\wedge\theta_{\bar{j}}. \end{cases} \tag{2.8}$$

Taking account of the coefficients of $\theta_\alpha\wedge\theta_\beta$ for each pair of indices α, β we get the following lemma.

Lemma 2.2. *Let M be a connected orientable hypersurface in CP^n. Denote by e the unit normal vector field on M and J the complex structure of CP^n. If $J(e)$ is a principal vector on M and the number of distinct principal curvatures is constant, then for any $p\in M$ there is a local orthonormal frame $\{e_A, e_{\bar{A}}=J(e_A)\}$ near p such that $e_{\bar{n}}=e$ and (2.1) are satisfied Furthermore, we have*

$$\lambda_n(\lambda_i+\lambda_{\bar{i}})=2(\lambda_i\lambda_{\bar{i}}-1), \tag{2.9}$$

$$d\lambda_n=0, \quad e_n(\lambda_\alpha)=0, \tag{2.10}$$

$$e_\beta(\lambda_\alpha)=(\lambda_\beta-\lambda_\alpha)\theta_{\alpha\beta}(e_\alpha), \tag{2.11}$$

$$(\lambda_j-\lambda_i)\theta_{ij}(e_n)=(\lambda_{\bar{j}}-\lambda_{\bar{i}})\theta_{\bar{i}\bar{j}}(e_n)=0, \tag{2.12}$$

$$(\lambda_{\bar{j}}-\lambda_i)\theta_{i\bar{j}}(e_n)=\lambda_n(\lambda_i-\lambda_{\bar{i}})\delta_{ij}/2, \tag{2.13}$$

$$(\lambda_\alpha-\lambda_\beta)\theta_{\beta\alpha}(e_\gamma)=(\lambda_\alpha-\lambda_\gamma)\theta_{\gamma\alpha}(e_\beta)=(\lambda_\gamma-\lambda_\beta)\theta_{\beta\gamma}(e_\alpha), \tag{2.14}$$

where α, β, γ are distinct and $\neq n$.

Remarks. (i) From (2.9) we see that $\lambda_i=\lambda_j$ if and only if $\lambda_{\bar{i}}=\lambda_{\bar{j}}$, $\lambda_i=\lambda_{\bar{j}}$ if and only if $\lambda_{\bar{i}}=\lambda_j$. Thus by choosing frame if necessary, we can assume that if $\lambda_i\neq\lambda_{\bar{i}}$ for some fixed i, then $\lambda_i\neq\lambda_{\bar{j}}$, $\lambda_{\bar{i}}\neq\lambda_j$ for all j.

(ii) Since λ_n is constant, λ_i is constant if and only if $\lambda_{\bar{i}}$ is constant by (2.9).

Lemma 2.3. *Under the assumptions of Lemma 2.2, if λ_i is constant and $\lambda_i\neq\lambda_{\bar{i}}$, then $\lambda_n\neq 0$ and λ_i, $\lambda_{\bar{i}}$ are the two distinct roots of equation*

$$y^2+\frac{4}{\lambda_n}y-1=0.$$

Proof Making use of the convention of Remark (i), by (2.11), (2.14) and (2.2) we have

$$\begin{cases}\theta_{i\bar{i}}(e_i)=\theta_{i\bar{i}}(e_{\bar{i}})=\theta_{\bar{i}\bar{i}}(e_i)=\theta_{\bar{i}j}(e_{\bar{i}})=0, & j\neq i,\\ \theta_{i\bar{i}}(e_j)=\theta_{i\bar{i}}(e_{\bar{j}})=0.\end{cases} \tag{2.15}$$

From (2.13) we get

$$\theta_{i\bar{i}}=\sum_{\alpha}\theta_{i\bar{i}}(e_\alpha)\theta_\alpha=-\frac{\lambda_n}{2}\theta_n. \tag{2.16}$$

Taking exterior differentiation of (2.16) and using (2.3) (2.7) we have

$$2\Big(-\sum_j\theta_{ij}\wedge\theta_{j\bar{i}}+(1+\lambda_i\lambda_{\bar{i}})\theta_i\wedge\theta_{\bar{i}}+\sum\theta_j\wedge\theta_{\bar{j}}\Big)=-\frac{\lambda_n}{2}\sum(\lambda_j+\lambda_{\bar{j}})\theta_j\wedge\theta_{\bar{j}}.$$

Taking account of the coefficients of $\theta_i\wedge\theta^{\bar{i}}$ and noting (2.15), we get

$$2(2+\lambda_i\lambda_{\bar{i}})=-\frac{\lambda_n}{2}(\lambda_i+\lambda_{\bar{i}}).$$

From this together with (2.9) we see $\lambda_n\neq0$ and λ_i, $\lambda_{\bar{i}}$ are the two distinct roots of

$$y^2+\frac{4}{\lambda_n}y-1=0.$$

Let $m(\lambda_\alpha)$ denote the multiplicity of λ_α. Then we have the following lemma.

Lemma 2.4. *Under the assumptions of Lemma 2.2, $m(\lambda_n)=1$.*

Proof Suppose $m(\lambda_n)>1$. Then $\lambda_i=\lambda_n$ for some i. But $\lambda_{\bar{i}}=\lambda_n+\frac{2}{\lambda_n}\neq\lambda_i$ by (2.9) and $\lambda_i=\lambda_n$ is constant. Thus $\lambda_n=\lambda_i$ satisfies the equation $y^2+\frac{4}{\lambda_n}y-1=0$ by Lemma 2.3, and we get $\lambda_n^2+3=0$, a contradiction.

Proof of Theorem A Without loss of generality we may assume M is connected. Therefore the conditions of Lemma 2.2 are satisfied by Prop. 1.2. Set $\lambda_n=2\operatorname{ctg}(2t)$. Then the roots of $x^2-\lambda_nx-1=0$ are $x_1=\operatorname{ctg}t$ and $x_2=\operatorname{ctg}\left(t-\frac{1}{2}\pi\right)$; when $\lambda_n\neq0$ the roots of $y^2+\frac{4}{\lambda_n}y-1=0$ are $y_1=\operatorname{ctg}\left(t-\frac{1}{4}\pi\right)$ and $y_2=\operatorname{ctg}\left(t-\frac{3}{4}\pi\right)$. It is clear that x_1, x_2, y_1, y_2 are distinct. If $\lambda_i=\lambda_{\bar{i}}$, then by (2.9) $\lambda_i=x_1$ or x_2; if $\lambda_i\neq\lambda_{\bar{i}}$, then by Lemma 2.3 $\lambda_i=y_1$ and $\lambda_{\bar{i}}=y_2$. We shall divide into four cases.

(a) All $\lambda_i=\lambda_{\bar{i}}=x_1$ (or x_2). M has 2 distinct principal curvatures: x_1 (resp. x_2), λ_n. $m(x_1)=2n-2$, $m(\lambda_n)=1$. By (1.2) we know that $\overline{M}=\pi^{-1}(M)$ has 2 distinct principal curvatures: x_1, x_2, $m(x_1)=2n-1$, $m(x_2)=1$ (or $m(x_1)=1$, $m(x_2)=2n-1$). $\overline{M}=S^{2n-1}(\sin t)\times S^1(\cos t)$.

(b) $\lambda_a=\lambda_{\bar{a}}=x_1$, $\lambda_r=\lambda_{\bar{r}}=x_2$, $1\leqslant a\leqslant p<r\leqslant p+q=n-1$. M has 3 distinct principal curvatures: x_1, x_2, λ_n. $m(x_1)=2p$, $m(x_2)=2q$, $m(\lambda_n)=1$. Similarly

$$\overline{M}=S^{2p+1}(\sin t)\times S^{2q+1}(\cos t).$$

(c) $\lambda_i\neq\lambda_{\bar{i}}$ for all i. M has 3 distinct principal curvatures: y_1, y_2, λ_n. $m(y_1)=m(y_2)=n-1$, $m(\lambda_n)=1$. $\overline{M}$ has 4 distinct principal curvatures: y_1, y_2, x_1, x_2. $m(y_1)=m(y_2)=n-1$, $m(x_1)=m(x_2)=1$. From [6] we know $\overline{M}=\overline{M}(2n, t)$.

(d) $\lambda_a\neq\lambda_{\bar{a}}$, $\lambda_r=\lambda_{\bar{r}}=x_1$, $\lambda_u=\lambda_{\bar{u}}=x_2$. $1\leqslant a\leqslant m<r\leqslant m+p<u\leqslant m+p+q=n-1$, $p+q>0$, p, $q\geqslant0$. We shall prove that $p=q$ and consequently M has 5 distinct principal curvatures: y_1, y_2, x_1, x_2, λ_n. $m(y_1)=m(y_2)=m$, $m(x_1)=m(x_2)=2p$, $m(\lambda_n)=1$.

From (1.2) we see that $\overline{M}$ has 4 distinct principal curvatures: y_1, y_2, x_1, x_2. The multiplicities of them are $m(y_1)=m(y_2)=m$, $m(x_1)=2p+1$, $m(x_2)=2q+1$ respectively. According to [2] we have $m(x_1)=m(x_2)$ and therefore $p=q$.

§3. Proof of Theorem B

Under the assumption of Lemma 2.2 we denote by $V(\lambda_\alpha)$ the distribution of the space of principal vectors corresponding to λ_α and discuss the integrability of $V(\lambda_\alpha)$ in this section to obtain Theorem B. Following the notations in section 2, we set $[\alpha]=\{\beta\mid\lambda_\beta=\lambda_\alpha\}$. Then $V(\lambda_\alpha)$ is spanned by $\{e_\beta\mid\beta\in[\alpha]\}$. If $V(\lambda_\alpha)+V(\lambda_\beta)$ denotes the direct sum distribution, then, noting the Remark (i) below Lemma 2.2, by (2.4)—(2.14) we can get easily the following lemmas.

Lemma 3.1. $[e_i, e_n]\in V(\lambda_i)+V(\lambda_{\bar{i}})$, $[e_{\bar{i}}, e_n]\in V(\lambda_i)+V(\lambda_{\bar{i}})$.

Lemma 3.2. *If* $\dim V(\lambda_i)>1$, *then*

$$e_j(\lambda_i)=e_{\bar{j}}(\lambda_i)=e_j(\lambda_{\bar{i}})=e_{\bar{j}}(\lambda_{\bar{i}})=0, \quad \textit{for any} \quad j\in[i].$$

The proofs of the following Theorems are straightforward. Only will we present argument for Theorem 3.5. The others will be left to the reader.

Theorem 3.3. *Suppose* $\dim V(\lambda_\alpha)=1$. *Then the integral curves of* $V(\lambda_\alpha)$ *are geodesics in* M *(resp. in* CP^n*) iff* λ_α *is constant (resp. vanishes).*

When $V(\lambda_\alpha)$ is completely integrable, we denote by $M(\lambda_\alpha)$ the integral submanifold. Then we have the following theorem.

Theorem 3.4. *Suppose* $\dim V(\lambda_i)>1$ *and* $\lambda_i\neq\lambda_{\bar{i}}$. *Then both* $V(\lambda_i)$ *and* $V(\lambda_{\bar{i}})$ *are completely integrable. Both* $M(\lambda_i)$ *and* $M(\lambda_{\bar{i}})$ *are totally umbilical submanifolds of* CP^n, *having constant curvature*

$$1+\lambda_i^2+\sum_{j\notin[i]}\left\{\left[\frac{e_j(\lambda_i)}{\lambda_j-\lambda_i}\right]^2+\left[\frac{e_{\bar{j}}(\lambda_i)}{\lambda_{\bar{j}}-\lambda_i}\right]^2\right\}$$

and

$$1+\lambda_{\bar{i}}^2+\sum_{j\notin[i]}\left\{\left[\frac{e_j(\lambda_{\bar{i}})}{\lambda_j-\lambda_{\bar{i}}}\right]^2+\left[\frac{e_{\bar{j}}(\lambda_{\bar{i}})}{\lambda_{\bar{j}}-\lambda_{\bar{i}}}\right]^2\right\}$$

respectively

Theorem 3.5. *Suppose* $\lambda_i=\lambda_{\bar{i}}$ *and consequently* $\dim V(\lambda_i)=2p\,(p\geq1)$. *Then* $V(\lambda_i)+V(\lambda_n)$ *is completely integrable. The integral submanifold* $M(\lambda_i, \lambda_n)$ *of* $V(\lambda_i)+V(\lambda_n)$ *is an open piece of* $M^c_{p,0}$ *which lies in a totally geodesic submanifold* CP^{p+1} *of* CP^n.

Proof Since $\lambda_i=\lambda_{\bar{i}}$, we have $[i]=[\bar{i}]$, $V(\lambda_i)=V(\lambda_{\bar{i}})$ and λ_i is constant by (2.9). By (2.11), (2.14) and (2.4) we get

$$\begin{cases}\theta_{ij}(e_k)=\theta_{ij}(e_{\bar{k}})=\theta_{\bar{i}j}(e_k)=\theta_{\bar{i}j}(e_{\bar{k}})=0, \\ \theta_{i\bar{j}}(e_k)=\theta_{i\bar{j}}(e_{\bar{k}})=\theta_{\bar{i}\bar{j}}(e_k)=\theta_{\bar{i}\bar{j}}(e_{\bar{k}})=0, \quad k\in[i],\ j\notin[i], \\ \theta_{in}(e_k)=\theta_{\bar{i}n}(e_{\bar{k}})=0,\ \theta_{in}(e_{\bar{k}})=-\theta_{\bar{i}n}(e_k)=-\lambda_i\delta_{ik}.\end{cases} \tag{3.1}$$

From this together with Lemma 3.1 we can see that $V(\lambda_i)+V(\lambda_n)$ is completely integrable.

On $M(\lambda_i, \lambda_n)$ we have $\theta_{\bar{n}}=0$, $\theta_j=\theta_{\bar{j}}=0$, $j\notin[i]$. Thus by (3.1), (2.4), (2.12) and (2.13) we get on $M(\lambda_i, \lambda_n)$

$$\theta_{ij}=\theta_{i\bar{j}}=\theta_{\bar{i}j}=\theta_{\bar{i}\bar{j}}=\theta_{nj}=\theta_{n\bar{j}}=\theta_{j\bar{n}}=\theta_{\bar{n}\bar{j}}=0, \quad \text{for} \quad j\notin[i]. \tag{3.2}$$

Let $\tilde{M}=\pi^{-1}(M(\lambda_i, \lambda_n))$, where π is the Hopf fibration. Then $\{\tilde{e}_0, \tilde{e}_n, \tilde{e}_k, \tilde{e}_{\bar{k}}|k\in[i]\}$ and $\{\tilde{e}_{\bar{n}}, \tilde{e}_j, \tilde{e}_{\bar{j}}|j\notin[i]\}$ are local orthonomal frame of tangent bundle $T\tilde{M}$ and normal bundle $N\tilde{M}$ of $\tilde{M}$ respectively. Equivalently on $\tilde{M}$ $\tilde{\theta}_j=\tilde{\theta}_{\bar{j}}=\tilde{\theta}_{\bar{n}}=0$ for $j\notin[i]$. Therefore by (1.2) and (3.2) we get on $\tilde{M}$

$$\tilde{\theta}_{\bar{n}j}=\tilde{\theta}_{\bar{n}\bar{j}}=0, \quad \text{for} \quad j\notin[i], \tag{3.3}$$

$$\tilde{\theta}_{ij}=\tilde{\theta}_{i\bar{j}}=\tilde{\theta}_{\bar{i}j}=\tilde{\theta}_{\bar{i}\bar{j}}=\tilde{\theta}_{nj}=\tilde{\theta}_{n\bar{j}}=\tilde{\theta}_{0j}=\tilde{\theta}_{0\bar{j}}=0, \quad \text{for} \quad j\notin[i]. \tag{3.4}$$

(3.4) shows that M is totally geodesic with respect to normal subbundle N_1 spanned by $\{\tilde{e}_j, \tilde{e}_{\bar{j}}|j\notin[i]\}$. (3.3) shows that N_1 is parallel in $N\tilde{M}$. By Theorem 1 of [10] $\tilde{M}$ lies in a totally geodesic submanifold S^{2p+3} of S^{2n+1} with N_1 perpendicular to S^{2p+3} everywhere. Thus $\tilde{e}_0$ is tangent to S^{2p+3} and the restriction of π on S^{2p+3} is compatible with π. Therefore $\pi(S^{2p+3})=CP^{p+1}$ is a totally geodesic submanifold of CP^n. Consequently $M(\lambda_i, \lambda_n)$ is a hypersurface in this CP^{p+1} with 2 distinct constant principal curvatures. Using the result of [4] we have the Theorem.

Theorem 3. 6. *Suppose $\lambda_i\neq\lambda_{\bar{i}}$ and* dim $V(\lambda_i)=m$. *If λ_i is constant, then $V(\lambda_i)+V(\lambda_{\bar{i}})+V(\lambda_n)$ is completely integrable. The integral submanifold $M(\lambda_i, \lambda_{\bar{i}}, \lambda_n)$ is an open piece of $M(2m+1, t)$ which lies in a totally geodesic submanifold CP^{m+1} of CP^n.*

Applying Theorems 3.3, 3.5 and 3.6 to the case (d) in the proof of Theorem A, we get Theorem B.

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