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A CORONA THEOREM FOR COUNTABLY MANY FUNCTIONS ON A CLASS OF INFINITELY CONNE CTED DOMAINS

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Abstract

Let D be a domain in the complex plane, and $H^\infty(D)$ be the Banach algebra of bounded analytic functions on D. Rosenblum proved a corona theorem for countably many functions on the open unit disk, Rudol extended the result to the finitely connected domains. In this paper the author uses Behrens' idea to extend the result to a class of infinitely connected domains.

§1. Introduction

Let D be a domain in the complex plane, and let $H^\infty(D)$ be the algebra of bounded analytic functions on D. Let $\mathscr{M}(D)$ be the maximal ideal space of $H^\infty(D)$, the domain can be identified with an open subset of $\mathscr{M}(D)$, by identifying a point $\lambda\in D$ with the homomorphism evaluated at λ. The corona problem asks whether D is dense in $\mathscr{M}(D)$. Carleson[2] proved that the open unit disk Δ is dense in $\mathscr{M}(\Delta)$. Stout[6] and others extended Carleson's theorem to finitely connected plannar domains, and, more generally to finite open Riemann surfaces. It is well know that D is dense in $\mathscr{M}(D)$ if and only if the following condition holds: If $f_1, \cdots, f_n\in H^\infty(D)$ and if $\inf\limits_{z\in D}\sum\limits_{j=1}^{n}|f_j(z)|^2>0$, then there exist $g_1, \cdots, g_n\in H^\infty(D)$ such that $f_1g_1+\cdots+f_ng_n=1$. Rosenblum[4] proved a corona theorem for countably many functions on the open unit disk Δ, he showed that if $\{a_j\}$ is a sequence of bounded analytic functions on Δ such that $\|\{a_j\}\|_\infty=\left(\sup\limits_{z\in\Delta}\sum\limits_{j=1}^{\infty}|a_j(z)|^2\right)^{1/2}<\infty$, then there exists a sequence $\{c_j\}$ of $H^\infty(\Delta)$ functions with $\|\{c_j\}\|_\infty<\infty$, and satisfying $\sum\limits_{j=1}^{\infty}a_jc_j=1$ on Δ if and only if $\delta=\inf\limits_{z\in\Delta}\sum\limits_{j=1}^{\infty}|a_j(z)|^2>0$. If $\delta>0$, he showed $\|\{c_j\}\|_\infty\leqslant 65\|\{a_j\}\|_\infty\delta^{-4}$. Rudol[5] extended the result to the finitely conneeted domains. In this paper we will use Behrens' idea[1] to extend the result to a class of infinitely connected domains.

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§2. Notations and Results

By Δ-domain we mean a domain D obtained from the open unit disk Δ by deleting the origin and a sequence of disjoint closed disks $\Delta_n=\Delta(c_n, r_n)=\{z: |z-c_n| \leq r_n\}$ with $c_n \to 0$. Throughout we assume that there exist numbers $R_n > r_n$ such that the disks $D_n=\Delta(c_n, R_n)$ are disjoint and such that $r_n/R_n \to 0$. Let $E_n(z)=r_n/(z-c_n)$ for $z \in \Delta_n^c = \mathbb{C} \setminus \Delta_n$, $n=1, 2, \cdots$. For notational convenience let $c_0=0$, $r_0=R_0=1$. $E_0(z)=z$. Choose a sequence of positive integers $\{k_n\}$ such that $\sum_n (r_n/R_n)^{k_n} < \infty$. Let $L_n(z)=(E_n(z))^{k_n}-(E_n(0))^{k_n}$. $L(z)=\sum_n L_n(z)$.

Let $H^\infty(\Delta \times N)$ be the class of bounded functions which are analytic on each slice of $\Delta \times N$, and $\mathcal{M}(\Delta \times N)$ its maximal ideal space. Let $X=\mathcal{M}(\Delta \times N) \setminus \bigcup_n \mathcal{M}(\Delta) \times \{n\}$, see [1] for details. Each $f \in H^\infty(D)$ can be written in $D_n \setminus \Delta_n$ as

$$f(z)=(P_n f)(z)+a_n(f)+F_n(z),$$

where

$$(P_n f)(z)=\frac{1}{2\pi i}\int_{\partial\Delta_n} \frac{f(\zeta)d\zeta}{\zeta-z}, \qquad a_n(f)=\frac{1}{2\pi i}\int_{\partial\Delta_n} \frac{f(\zeta)}{\zeta-c_n}\, d\zeta$$

and $F_n(z)$ is analytic in D_n, see [3] for more details. Let $\mathcal{M}_0$ denote the fiber in $\mathcal{M}(D)$ at origin and $A_0=H^\infty(D)|_{\mathcal{M}_0}$.

Let $\Omega=D$ or $\Delta \times N$, and $\varphi \in \mathcal{M}(\Omega)$, $\{f_j\}$ be a sequence of $H^\infty(\Omega)$ functions. We say that φ is weak continuous with respect to $\{f_j\}$, if for any sequence $\{g_j\}$ of $H^\infty(\Omega)$ functions, $\varphi(\sum_j f_j g_j)=\sum_j \varphi(f_j)\varphi(g_j)$, and that φ is uniformly adherent to Ω, if $\inf_{z \in \Omega} \sum_j |f_j(z)-\varphi(f_j)|^2=0$ for all $\{f_j\}$ which φ is weak continuous with respect to. We say that Ω is uniformly dense in $\mathcal{M}(\Omega)$ if every $\varphi \in \mathcal{M}(\Omega)$ is uniformly adherent to Ω. It is easy to see that uniformly denseity implies denseity.

Theorem 1. *Let D be a Δ-domain with $r_n/R_n \to 0$. If $\varphi \in \mathcal{M}_0$* and $\hat{L}(\varphi) \neq 0$, *then φ is uniformly adherent to D.*

Theorem 2. *Let D be a Δ-domain. Suppose there exists a positive integer m such that $\sum_n (r_n/R_n)^m < \infty$. Let $\varphi \in \mathcal{M}_0$. If for some $f \in H^\infty(D)$, $\varphi(f) \neq 0$ but $a_n(f) \to 0$, then φ is uniformly adherent to D.*

Theorem 3. *Let D be a domain in Theorem 1 or 2, and $\{a_n(f)\}$ is a Cauchy sequence of l^2 for each $f \in H^\infty(D)$. Then D is uniformly dense in $\mathcal{M}(D)$.*

Theorem 4. *Let D be a domain in Theorem 3. If $\{f_j\}$ is a sequence of $H^\infty(D)$ functions such that $\|\{f_j\}\|_\infty < \infty$, $\delta=\inf_{z \in D} \sum_{j=1}^{\infty} |f_j(z)|^2 > 0$, then there exists a sequence of $H^\infty(D)$ functions $\{g_j\}$, $\|\{g_j\}\|_\infty < \infty$, such that*

$$\sum_{j=1}^{\infty} f_j g_j = 1.$$

§ 3. Preliminary Lemmas and Proofs

Let $\{f_j\}$ be a sequence of $H^\infty(D)$ functions, $\|\{f_j\}\|_\infty < \infty$. We set

$$J = J(\{f_j\}) = \left\{\sum_{j=1}^{\infty} f_j g_j: \{g_j\} \text{ is a sequence of } H^\infty(D) \text{ functions}, \ \|\{g_j\}\|_\infty < \infty\right\}.$$

It is easy to see that J is an ideal of $H^\infty(D)$.

Lemma 1. *Let Ω be any domain or $\Delta \times N$. Then Ω is uniformly dense in $\mathscr{M}(\Omega)$ if and ony if the following condition holds: If $\{f_j\}$ is a sequence of $H^\infty(\Omega)$ functions, $\|\{f_j\}\|_\infty < \infty$ and $\delta = \inf_{z \in \Omega} \sum_{j=1}^{\infty} |f_j(z)|^2 > 0$, then there exists a sequence of $H^\infty(\Omega)$ functions $\{g_j\}$, $\|\{g_j\}\|_\infty < \infty$, such that $\sum_{j=1}^{\infty} f_j g_j = 1$.*

Proof Suppose Ω is uniformly dense in $\mathscr{M}(\Omega)$, the condition $\inf_{z \in \Omega} \sum_{j=1}^{\infty} |f_j(z)|^2 > 0$ ismplies the ideal J is not proper. Otherwise, there exists a maximal ideal $m \supset J \supset \{f_j\}$, thus

$$0 < \inf_{z \in \Omega} \sum_{j=1}^{\infty} |f_j(z)|^2 = \inf_{z \in \Omega} \sum_{j=1}^{\infty} |f_j(z) - m(f_j)|^2, \tag{1}$$

m is weak continuous with respect to $\{f_j\}$, and hence the right side of (1) is zero by uniformly density. This is a contradiction.

Conversely, suppose Ω is not uniformly dense in $\mathscr{M}(\Omega)$. Then some point $m_0 \in \mathscr{M}(\Omega)$ is weak continuous with respect to $\{f_j\}$ such that $\inf_{\Omega} \sum_{j=1}^{\infty} |f_j - m_0(f_j)|^2 > 0$. It is easy to see that $\sum_{j=1}^{\infty} |m_0(f_j)|^2 \le \|\{f_j\}\|_\infty^2$. Let $g_j = f_j - m_0(f_j)$, then $\|\{g_j\}\|_\infty < \infty$, $\inf_{\Omega} \sum_{j=1}^{\infty} |g_j|^2 > 0$, and there is a sequence of $H^\infty(\Omega)$ functions $\{h_j\}$, $\|\{h_j\}\|_\infty < \infty$ such that $\sum_{j=1}^{\infty} g_j h_j = 1$. m_0 is weak continuous with respect to $\{g_j\}$ and we have

$$1 = m_0(1) = \sum_{j=1}^{\infty} m_0(g_j) m_0(h_j) = 0,$$

a contradiction.

Lemma 2. *$\Delta \times N$ is uniformly dense in $\mathscr{M}(\Delta \times N)$.*

Proof Suppose $\{f_j\}$ is a sequence of $H^\infty(\Delta \times N)$ functions with

$$\|\{f_j\}\|_\infty = (\sup_{\Delta \times N} \sum_{j=1}^{\infty} |f_j(z, n)|^2)^{1/2} < \infty, \quad \delta = \inf_{\Delta \times N} \sum_{j=1}^{\infty} |f_j(z, n)|^2 > 0.$$

By Rosenblum's solution of the corona theorem for countably many functions on open unit disk Δ, there is a sequence of functions $\{g_{jm}\}$ of $H^\infty(\Delta \times \{m\})$ $\|\{g_{jm}\}\|_\infty \le M$, where M depends only on δ, such that $\sum_{j=1}^{\infty} f_j(z, n) g_{jm}(z) = 1$. Then $\{g_{jm}\}$ determines

$\{g_j\}\subset H^\infty(\Delta\times N)$ with $\|\{g_j\}\|^\infty\leqslant M$, and then

$$\sum_{j=1}^\infty f_j(z,\ n)g_j(z,\ n)=1 \text{ for all } (z,\ n)\in\Delta\times N.$$

By Lemma 1, $\Delta\times N$ is uniformly dense in $\mathscr{M}(\Delta\times N)$.

$$\Psi:\ H^\infty(D)\to H^\infty(\Delta\times N),$$

Define

$$\Psi(f)(z,\ n)=(P_nf)\cdot E_n^{-1}(z)+a_n(f),\quad f\in H^\infty(D).$$

Lemma 3. *Let D be a Δ-domain with $r_n/R_n\to 0$. Then Ψ induces a continuous algebra isometric isomorphism ρ of A_0 onto a closed subalgebra B_0 of $B=H^\infty(\Delta\times N)|_X$ as indicated by the following diagram*

$$\begin{array}{ccc} H^\infty(D) & \xrightarrow{\ \Psi\ } & H^\infty(\Delta\times N) \\ \text{restriction}\ \big\downarrow & & \big\downarrow\ \text{restriction} \\ A_0 & \xrightarrow{\ \rho\ } & B \end{array}$$

Proof See [3].

The map θ: $\mathscr{M}(B_0)\to\mathscr{M}_0$ defined by $\theta(\varphi)(f)=\varphi(\rho(f))$ for $\varphi\in\mathscr{M}(B_0)$, $f\in A_0$, is a homomorphism. Note that

$$H^\infty(\Delta\times N)\cong ZH^\infty(\Delta\times N)+l^\infty,$$

where Z is the "coordinate" function $Z(\lambda,\ n)=\lambda$, l^∞ is the space of bounded sequences. The sequence $\{a_n(f)\}$ is the l^∞-component of $\Psi(f)$.

Lemma 4. *Let $T\in H^\infty(\Delta\times N)$ be defined by $T(z,\ n)=z^{k_n}$. Then $\rho(L)=T$ and*

$$TH^\infty(\Delta\times N)|_X+C\subset B_0.$$

If $\varphi\in\mathscr{M}(B_0)$ and $\varphi(T)\neq 0$, then φ extends to an evalution homomorphism at point of X.

Proof See [3].

Lemma 5. *Let $\psi\in\Psi(B_0)$ and $\varphi(T)\neq 0$, $T(z,\ n)=z^{k_n}$. Then the corres ponding homomorphism $\theta(\varphi)$ in M_0 is uniformly adherent to D.*

Proof Suppose $\varphi(T)\neq 0$. Then $\varphi\in X$ by Lemma 4. Let $\theta(\varphi)=\varphi^*$. By Lemma 2, X is uniformly adherent to $\Delta\times N$, that is

$$\inf_{\Delta\times N}\sum_{j=1}^\infty|\Psi(f_j)(z,\ n)-\varphi(\Psi(f_j))|^2=0 \tag{2}$$

only if φ is weak continuous with res pect to $\Psi(f_j)$. If we assume (2), for any $\varepsilon>0$, we can choose a net $(z_\alpha,\ n_\alpha)$ in $\Delta\times N$ with $n_\alpha\to\infty$ and $(z_\alpha,\ n_\alpha)$ converges to φ, such that for sufficiently large n_α

$$\sum_{j=1}^\infty|\Psi(f_j)(z_\alpha,\ n_\alpha)-\varphi(\Psi(f_j))|<\varepsilon.$$

Since $r_n/R_n\to 0$, then $E_{n_\alpha}^{-1}(z_\alpha)\in B_n\backslash\Delta_n$, for n_α sufficiently large, we have

$$\Big(\inf_D\sum_{j=1}^\infty|f_j-\varphi^*(f_j)|^2\Big)^{1/2}\leqslant\Big(\sum_{j=1}^\infty|\Psi(f_j)(z_\alpha,\ n_\alpha)-\varphi(\Psi(f_j)|^2\Big)^{1/2}$$
$$+\Big(\sum_{j=1}^\infty|f_j(E_{n_\alpha}^{-1}(z_\alpha))-P_{n_\alpha}f_j(E_{n_\alpha}^{-1}(z_\alpha))-a_{n_\alpha}(f_j)|^2\Big]^{1/2}. \tag{3}$$

The second term of the right side of (3) is less than $\|\{f_j\}\|_\infty \dfrac{\sqrt{r_{n_\alpha}}}{\sqrt{R_{n_\alpha}}-\sqrt{r_{n_\alpha}}}$ by [7], and hence φ^* is uniformly adherent to D.

Now we assume that φ^* is weak continuous with respect to $\{f_j\}$ Since $\varphi\in X$, by [7], we have

$$\varphi\Big(\sum_{j=1}^{\infty}\Psi(f_j)\Psi(g_j)\Big)=\sum_{j=1}^{\infty}\varphi(\Psi(f_j))\varphi(\Psi(g_j))$$

for $h_j(z, n)\in H^\infty(\Delta\times N)$, $\|\{h_j\}\|_\infty<\infty$. Since $Th_j\in B_0$, there is a sequence $\{g_j\}$ of $H^\infty(D)$, functions such that $\Psi(g_j)=Th_j$. Hence

$$\varphi\Big(\sum_{j=1}^{\infty}\Psi(f_j)h_j\Big)=\varphi\Big(\sum_{j=1}^{\infty}\Psi(f_j)Th_j\Big)/\varphi(T)=\varphi\Big(\sum_{j=1}^{\infty}\Psi(f_j)\Psi(g_j)\Big)/\varphi(T)$$
$$=\sum_{j=1}^{\infty}\varphi(\Psi(f_j))\varphi(h_j)$$

and this completes the proof.

Proof of Theorem 1 Let $\tilde{\varphi}=\theta^{-1}(\varphi)$. Then $\tilde{\varphi}(T)=\varphi(L)\neq 0$. By Lemma 4, $\tilde{\varphi}$ can be extended to an evaluation homomorphism at a point of X. Thus $\theta(\tilde{\varphi})=\varphi$ is uniformly adherent to D, by Lemma 5.

Proof of Theorem 2 Let $\theta^{-1}(\varphi)=\tilde{\varphi}$,

$$L(z)=\sum_n[((r_n/(z-c_n))^m-(r_n/(-c_n))^m]$$

and $T(z, n)=z^m$, so $T=\rho(L)=Z^m|_X$. By Lemma 4, $Z^mH^\infty(\Delta\times N)|_X\subset B_0$. Since $a_n(f)\to 0$, $\rho(f)\in ZH^\infty(\Delta\times N)|_X$ and $[\rho(f)]^m=\rho(f^m)\in Z^mH^\infty(\Delta\times N)|_X$. Now $0\neq\varphi(f^m)=\tilde{\varphi}(\rho(f^m))$. Thus $\tilde{\varphi}$ is not zero on $Z^mH^\infty(\Delta\times N)|_X$, $\varphi(L)=\tilde{\varphi}(T)\neq 0$ and φ is uniformly adherent to D by Theorem 1.

Proof of Theorem 3 Difine $\varphi_0(f)=\lim_n a_n(f)$, $f\in H^\infty(D)$. Then $\varphi_0\in\mathscr{M}_0$ by [3]. Let $\{f_j\}$ be a sequence of $H^\infty(D)$ functions with $\|\{f_j\}\|_\infty<\infty$. Let $z\in B_n\backslash\Delta_n$. Then

$$\Big(\sum_{j=1}^{\infty}|f_j(z)-a_n(f_j)|^2\Big)^{1/2}\leq\frac{1}{2\pi}\int_{\partial\Delta_n}\frac{\Big(\sum_{j=1}^{\infty}|f_j(\zeta)|^2\Big)^{1/2}}{|\zeta-z|}|d\zeta|\leq\|\{f_j\}\|_\infty\ r_n/(S_n-r_n)\to 0.$$

Thus

$$\Big(\inf_D\sum_{j=1}^{\infty}|f_j-\varphi_0(f_j)|^2\Big)^{1/2}\leq\|\{f_j\}\|_\infty r_n/(S_n-r_n)+\Big(\sum_{j=1}^{\infty}|a_n(f_j)-\varphi_0(f_j)|^2\Big)^{1/2}.$$

Hence φ_0 is uniformly adherent to D. If $\varphi\in\mathscr{M}_0\backslash\{\varphi_0\}$, Theorem 2 shows φ is uniformly adherent to D. If $\varphi\notin\mathscr{M}_0$, we assume that $\varphi(z)=z_0\in\partial\Delta_n$. Since $(f(z)-P_nf(z)-a_n(f)-F(z_0))/(z-z_0)$ is a function of $H^\infty(D)$ whenever $f\in H^\infty(D)$,

$$\Big(\inf_D\sum_{j=1}^{\infty}|f_j-\varphi(f_j)|^2\Big)^{1/2}\leq\inf_D\Big(\sum_{j=1}^{\infty}|P_nf_j-\varphi(P_nf_j)|^2\Big)^{1/2}+\Big(\sum_{j=1}^{\infty}|F_j(z)-F_j(z_0)|^2\Big)^{1/2},$$

the first term is zero by Rosenblum's result for open unit disk, and second term tends to zero by the analyticity of F_j on D_n. Hence φ is uniformly adherent to D.

Theorem 4 is a consequence of Lemma 1 and Theorem 3.

Remark. It would be interesting to find a domain D characterized only by the geometric conditions, such that D is dense in $\mathscr{M}(D)$ but D is not dense uniformly in $\mathscr{M}(D)$.

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